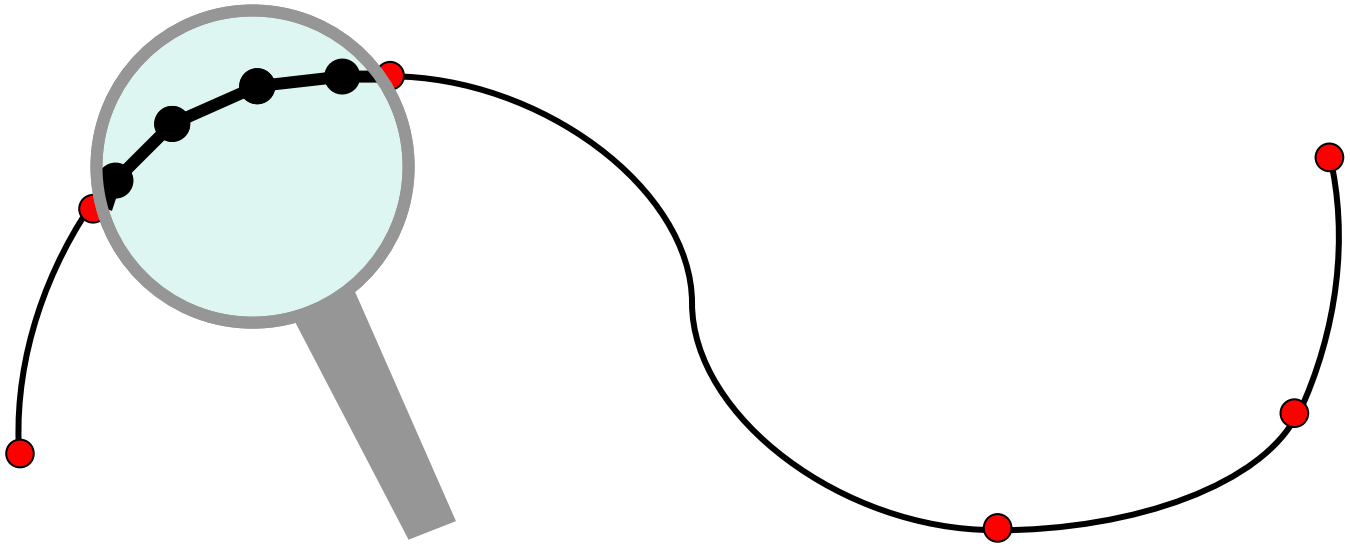
# Computergrafik SS 2014

Oliver Vornberger

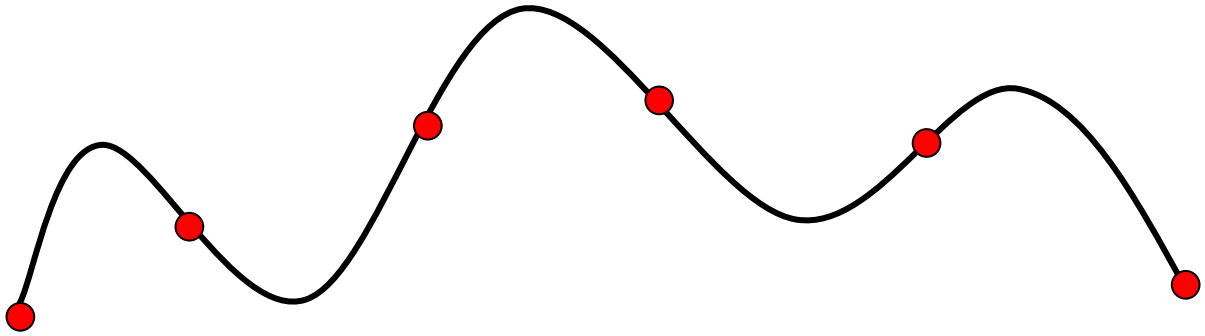
## Kapitel 7:
## 2D-Kurven

# Spezifikation einer Kurve



Stützpunkte $P_0$, $P_1$, …, $P_n$

# Algebraischer Ansatz

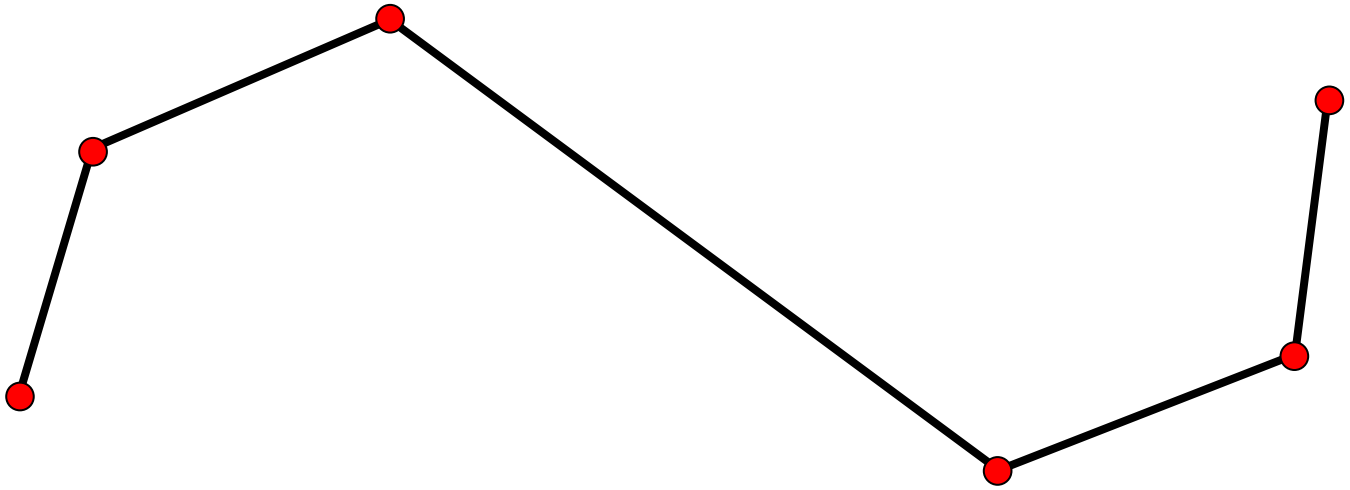


Bestimme n+1 Koeffizienten für Polynom n-ten Grades

$$y = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + ... + a_1 \cdot x + a_0$$

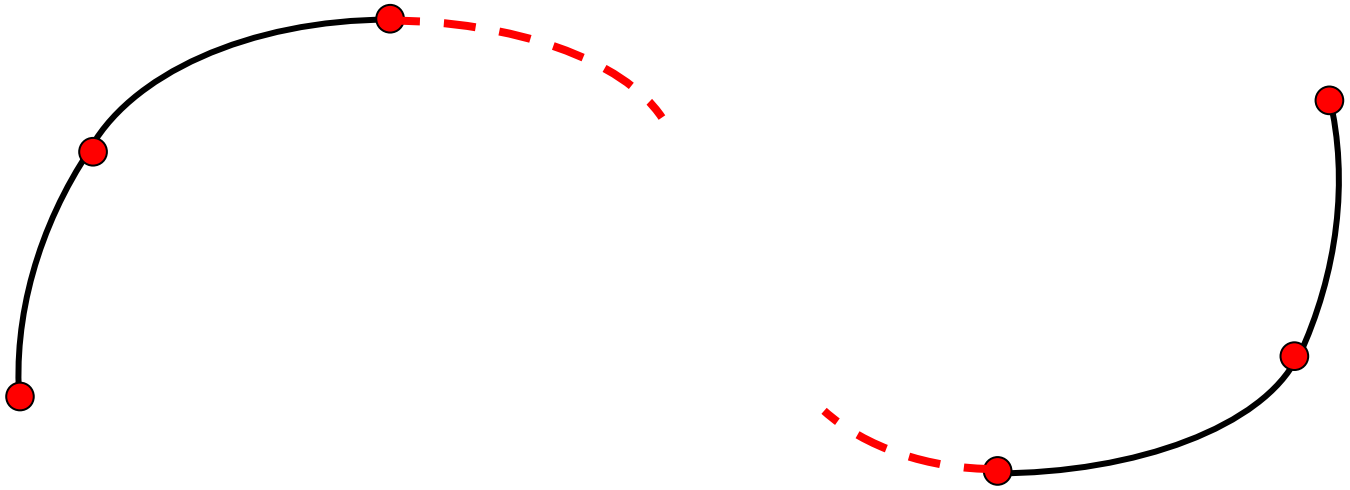
Oszillation! Rechenaufwand! Rundungsfehler !

# lineare Splines



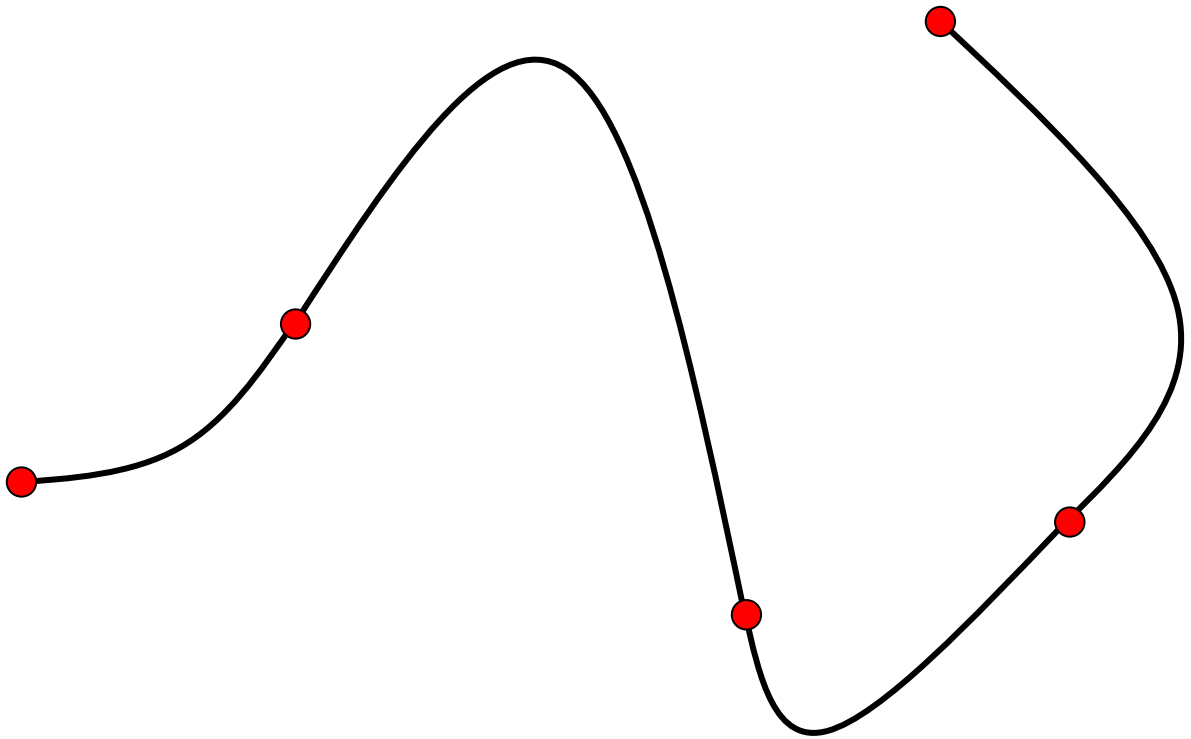
verbinde zwei aufeinanderfolgende Punkte
durch eine Gerade

# quadratische Splines

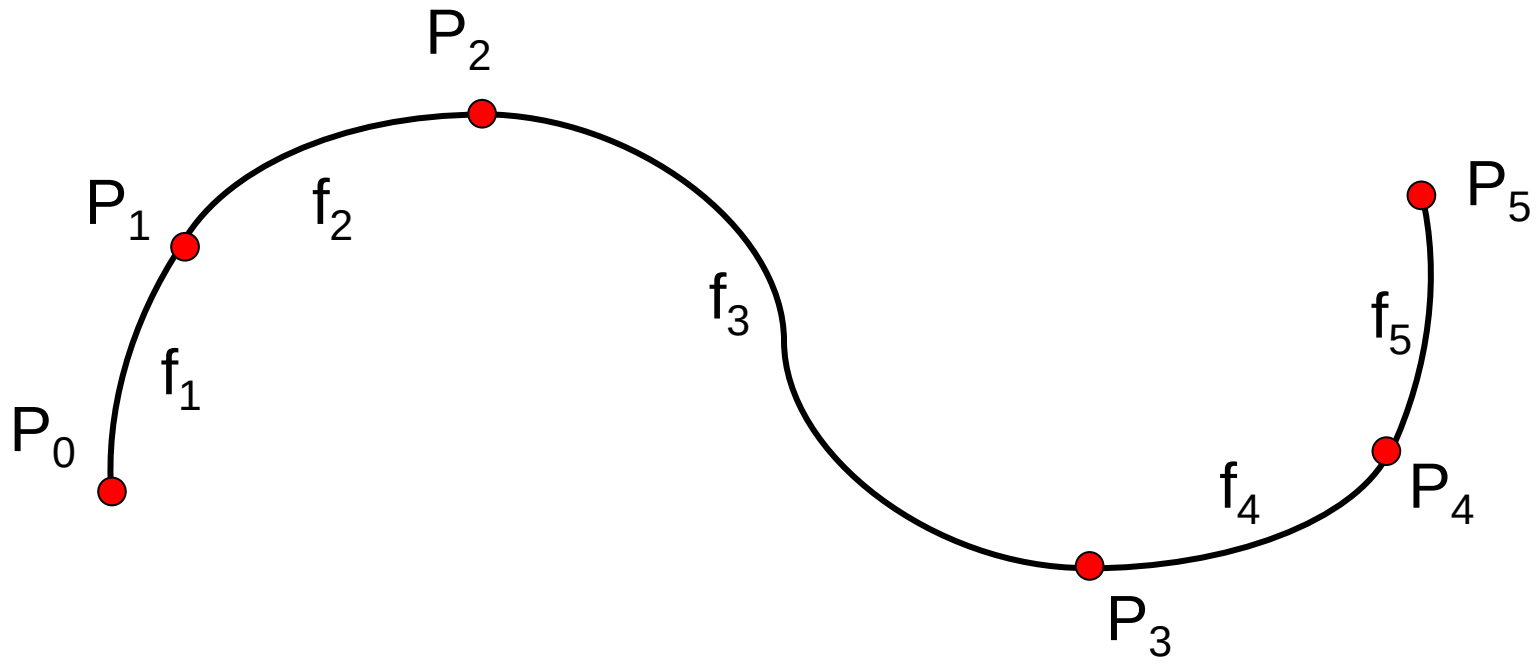


verbinde zwei aufeinanderfolgende Punkte
durch eine Kurve 2. Grades

# quadratische Splines

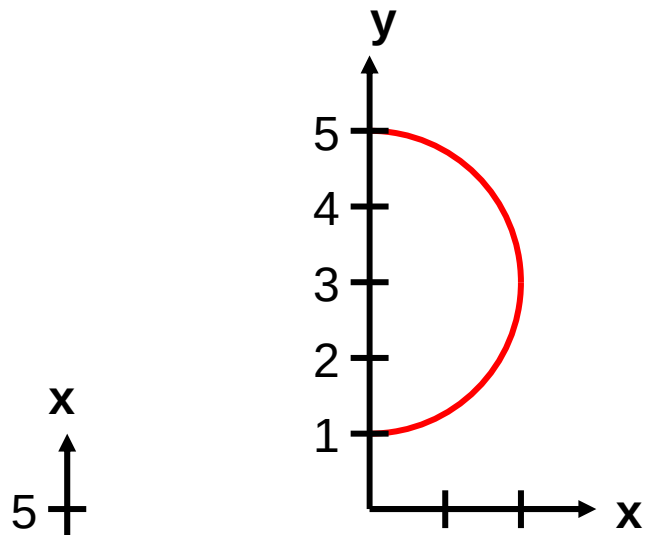


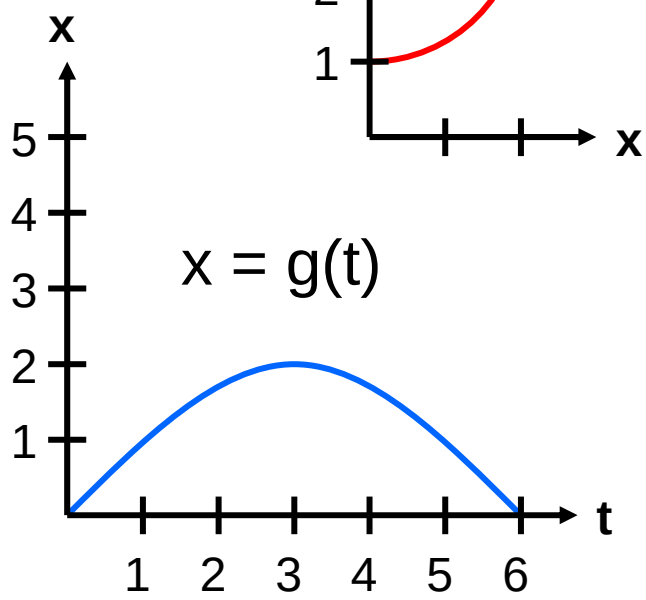
# kubische Splines



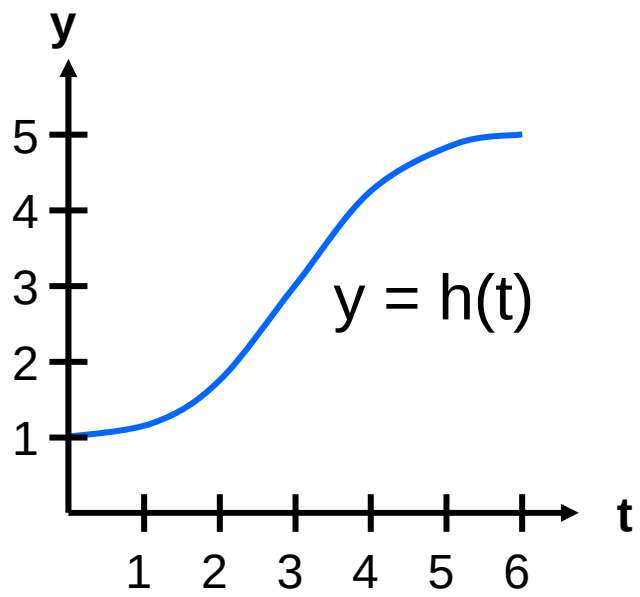

Verbinde zwei aufeinanderfolgende Punkte durch eine Kurve 3. Grades

# Parametrisierte Kurvengleichung

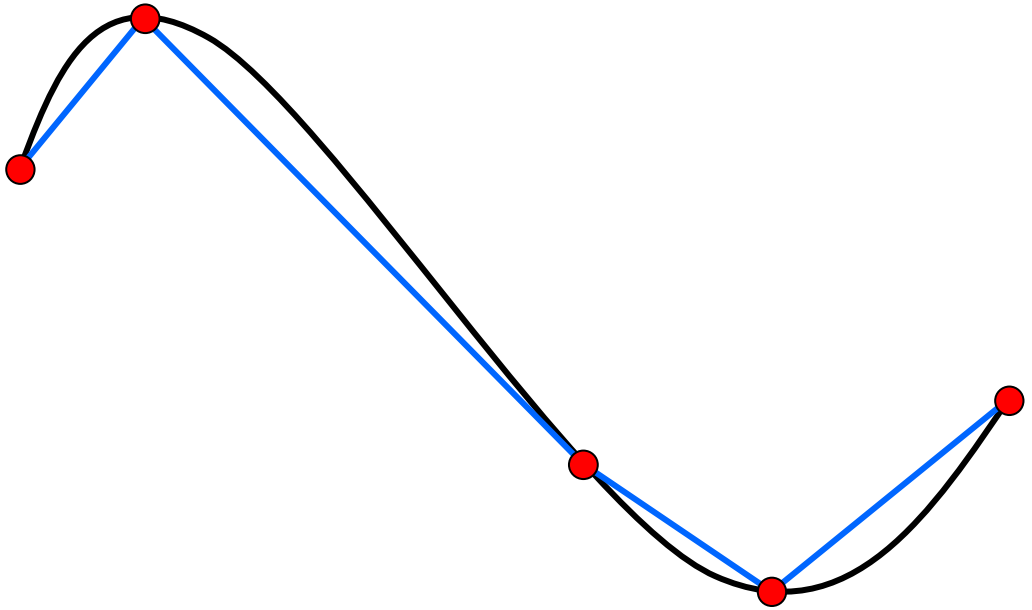



$$f(t) = [g(t), h(t)]$$



# Intervallgrenzen



$$\Delta_i = \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2}$$

# Kurvenabschnitte

Gesucht ist pro Intervall i, i=1, …, n

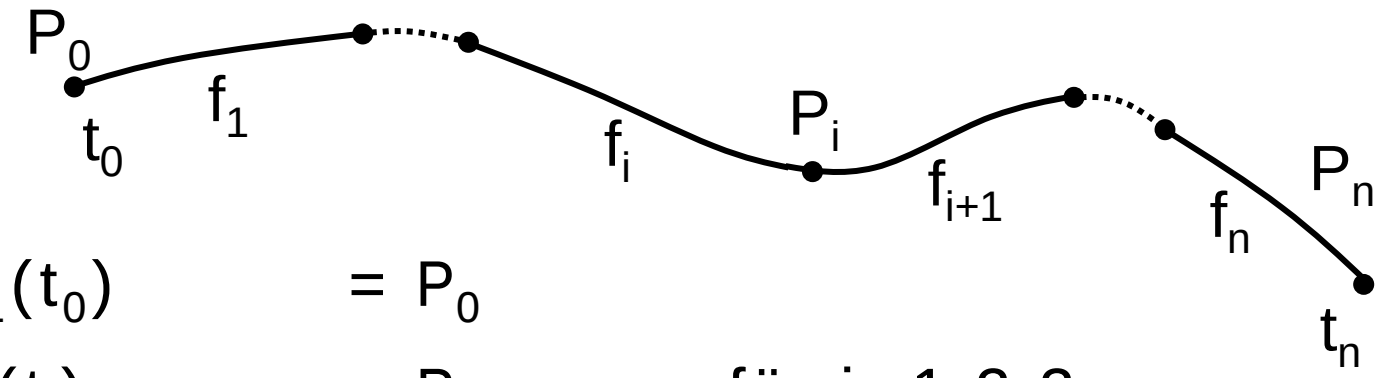
$$f_i(t) = a_i + b_i \cdot t + c_i \cdot t^2 + d_i \cdot t^3$$

genauer:

$$g_i(t) = a^g_i + b^g_i \cdot t + c^g_i \cdot t^2 + d^g_i \cdot t^3$$

$$h_i(t) = a^h_i + b^h_i \cdot t + c^h_i \cdot t^2 + d^h_i \cdot t^3$$

# Gleichungssystem




| | | |
|---|---|---|
| $f_1(t_0)$ | $= P_0$ | |
| $f_i(t_i)$ | $= P_i$ | für $i = 1,2,3,\ldots,n$ |
| $f_i(t_i)$ | $= f_{i+1}(t_i)$ | für $i = 1,2,3,\ldots,n-1$ |
| $f'_i(t_i)$ | $= f'_{i+1}(t_i)$ | für $i = 1,2,3,\ldots,n-1$ |
| $f''_i(t_i)$ | $= f''_{i+1}(t_i)$ | für $i = 1,2,3,\ldots,n-1$ |
| $f_1''(t_0)$ | $= 0$ | |
| $f_n''(t_n)$ | $= 0$ | |

$2 \cdot 4n$ Gleichungen mit
$2 \cdot 4n$ Unbekannten

# Approximation

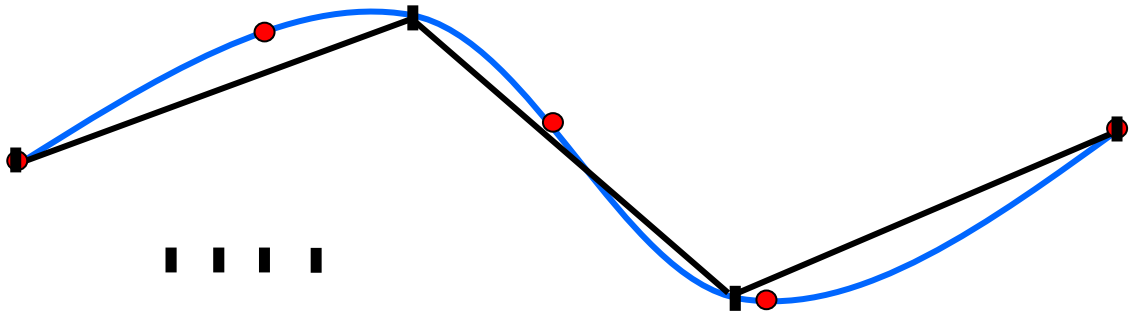
Gegeben n+1 Stützpunkte $P_0, P_1, \ldots, P_n$

Berechne Kurvenabschnitte $f_1, f_2, \ldots, f_n$

Bestimme Zahl der Interpolationspunkte k

Verteile längs der Kurvenabschnitte

Zeichne k-1 Geradenabschnitte



# Java-Applet zu kubischen Splines

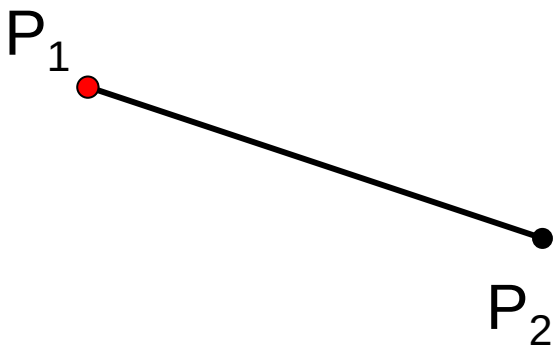
~cg/2014/skript/Applets/Splines/App.html

# Bewertung von Splines

bei vielen Stützpunkten:

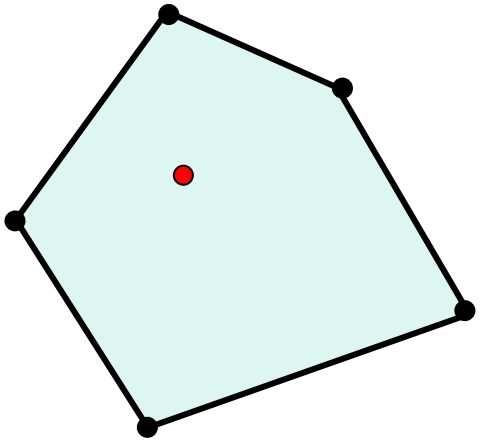
- hoher Rechenaufwand zur Lösung des Gleichungssystems
- Kein lokaler Einfluss möglich

# Kontrollpunkte




$$P = (1-t) \cdot P_1 + t \cdot P_2$$

**1.0000**

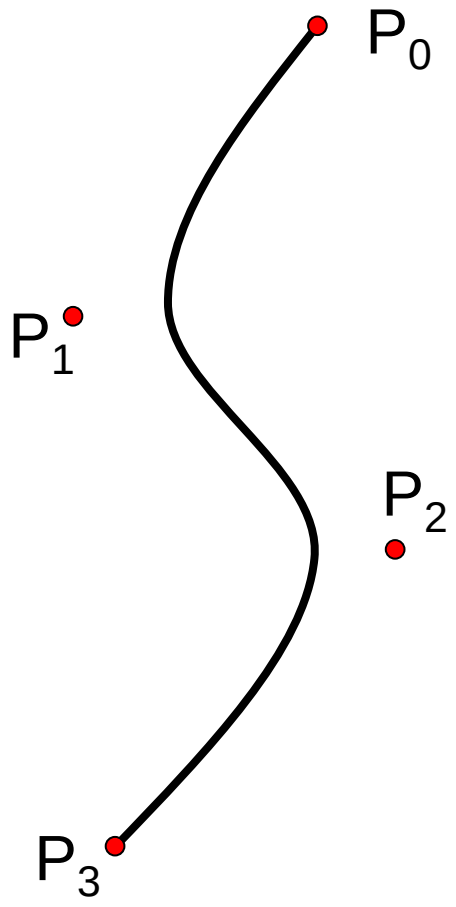


$$P = \sum_{i=0}^{n} m_i \cdot P_i$$

# Bézier-Kurven

Pierre Bézier
1911-1999
Ingenieur bei Renault

Paul deCasteljau
1930 -
Ingenieur bei Citroen



# Bernstein-Polynome

*n+1* Kontrollpunkte

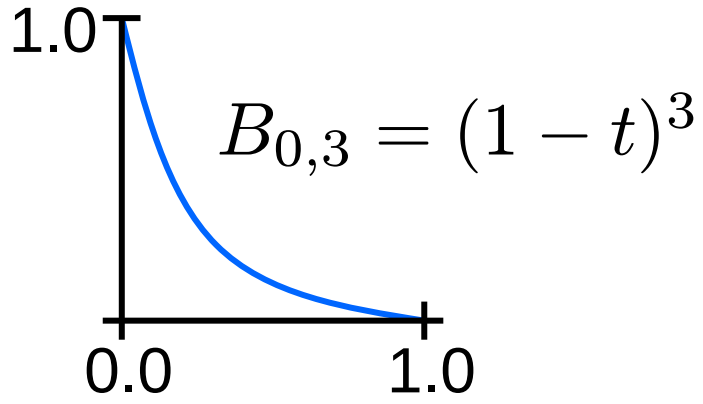
Jeder Kontrollpunkt $P_i$
wird gewichtet mit Bernstein-Polynom $B_i$

$$P(t) = \sum_{i=0}^{n} B_{i,n}(t) \cdot P_i, \quad 0 \leq t \leq 1$$

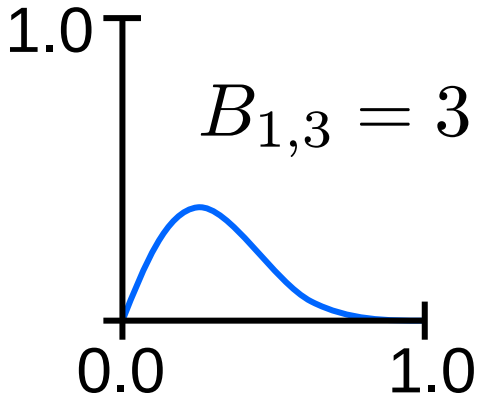
$$B_{i,n}(t) = \begin{pmatrix} n \\ i \end{pmatrix} \cdot t^i \cdot (1-t)^{n-i}, \; i = 0, ..., n$$

$$= \frac{n!}{i! \cdot (n-i)!} \cdot t^i \cdot (1-t)^{n-i}$$

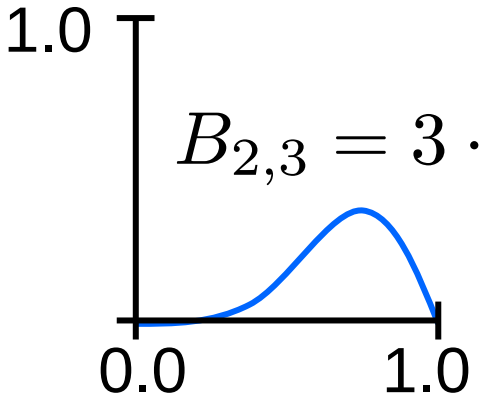
# Bernstein-Polynome 3. Grades



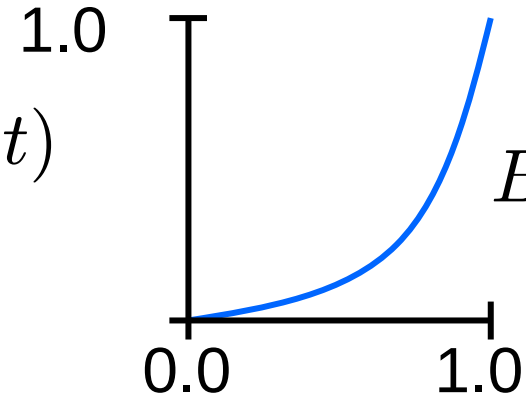

$$B_{0,3} = (1-t)^3$$




$$B_{1,3} = 3 \cdot t \cdot (1-t)^2$$

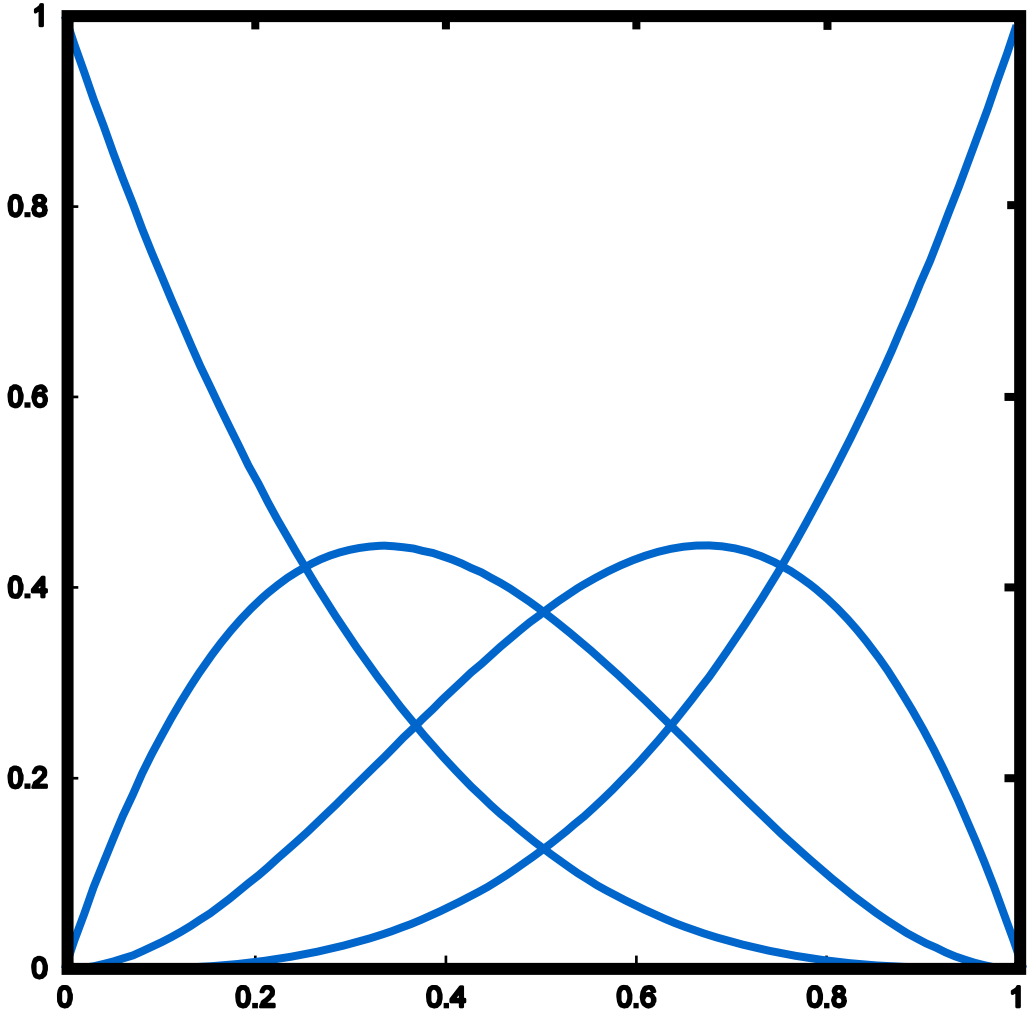



$$B_{2,3} = 3 \cdot t^2 \cdot (1-t)$$

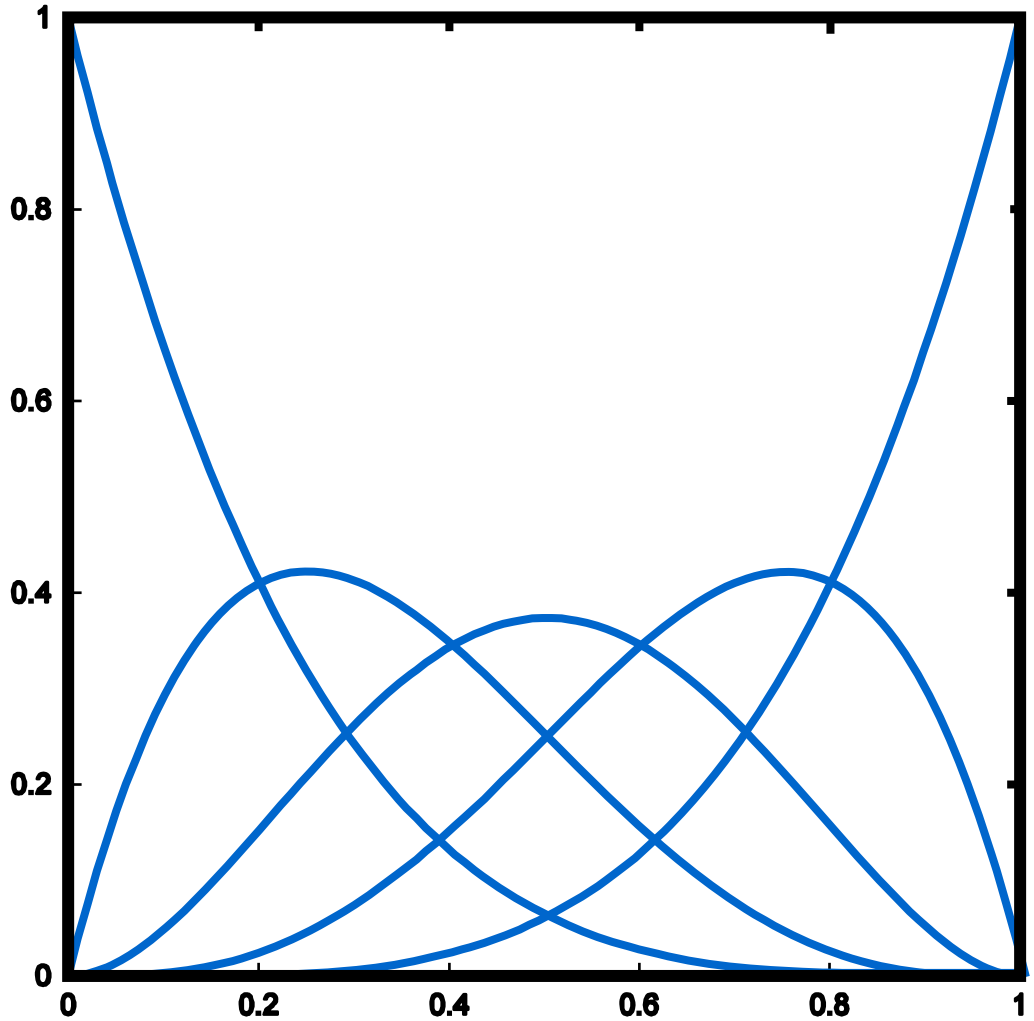



$$B_{3,3} = t^3$$

# Bernsteinpolynome 3. Grades



# Bernsteinpolynome 4. Grades



# Eigenschaften der Bernstein-Polynome

Alle Bernsteinpolynome sind positiv auf [0,1]

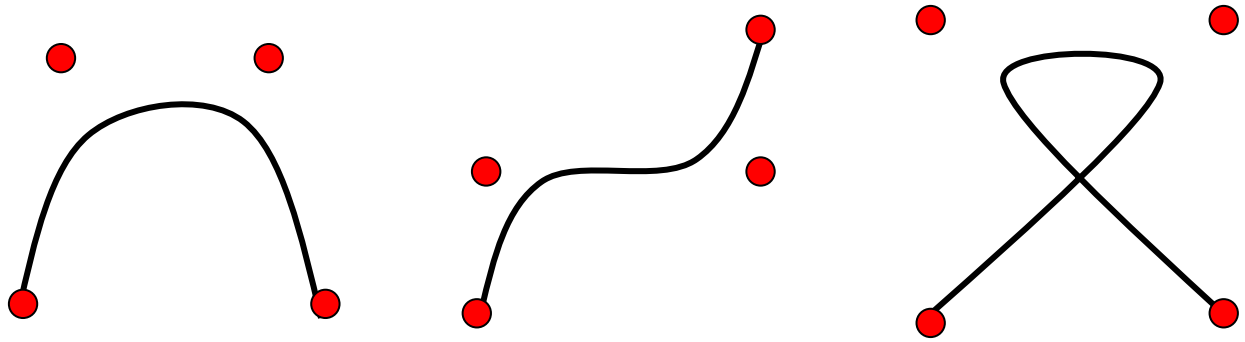
$$\sum_{i=0}^{n} B_{i,n}(t) = 1 \qquad \text{für jedes feste t}$$

$$B_{i,n}(t) = B_{n-i,n}(1-t)$$

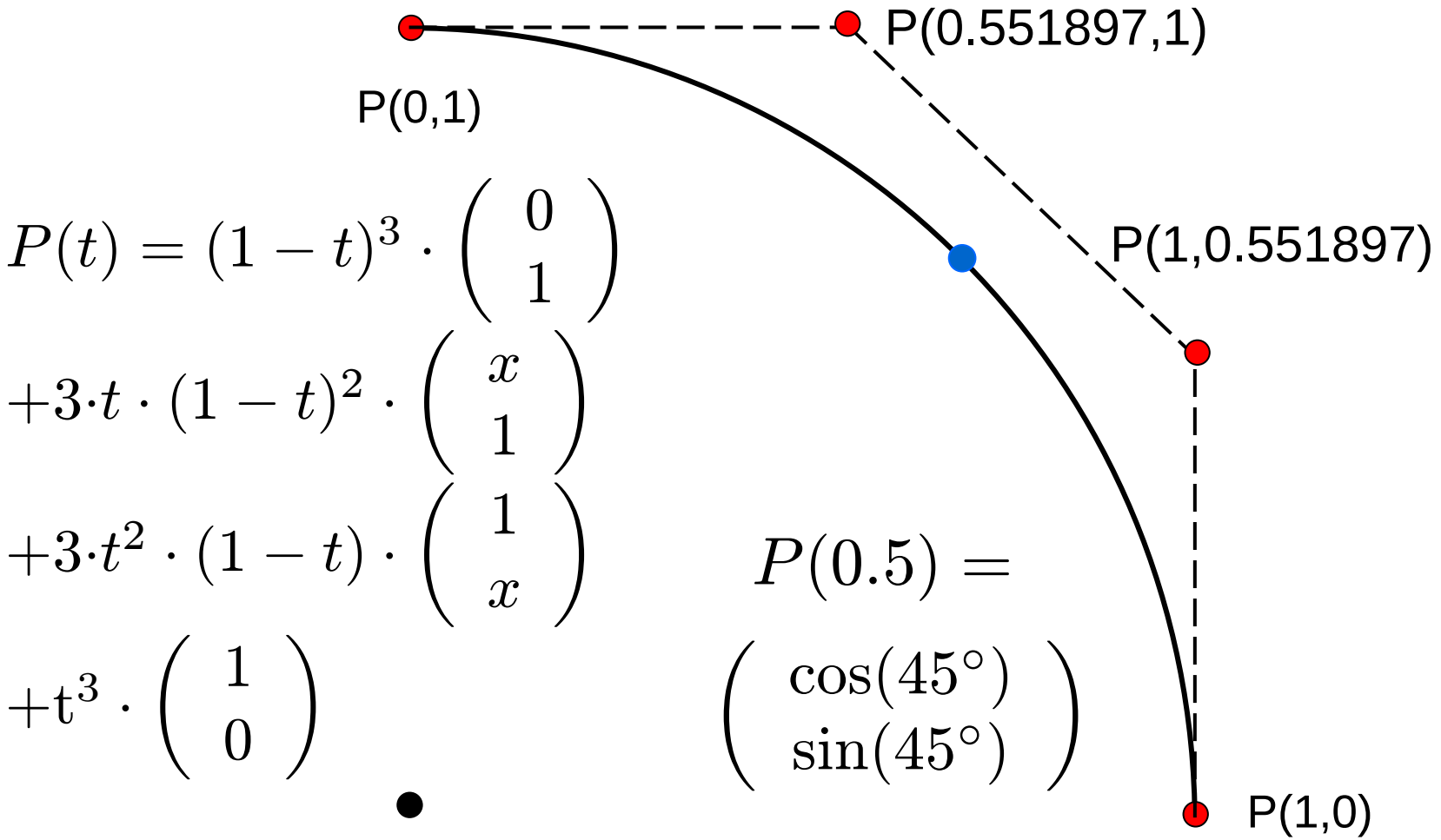
$$\text{Maxima von } B_{i,n}(t) \text{ bei } t = \frac{i}{n}, i = 0, \ldots, n$$

$$\frac{dB_{i,n}(t)}{dt} = n \cdot (B_{i-1,n-1}(t) - B_{i,n-1}(t)), i > 0$$

# Beispiele für Bézierkurven



# Approximation von Viertelkreis



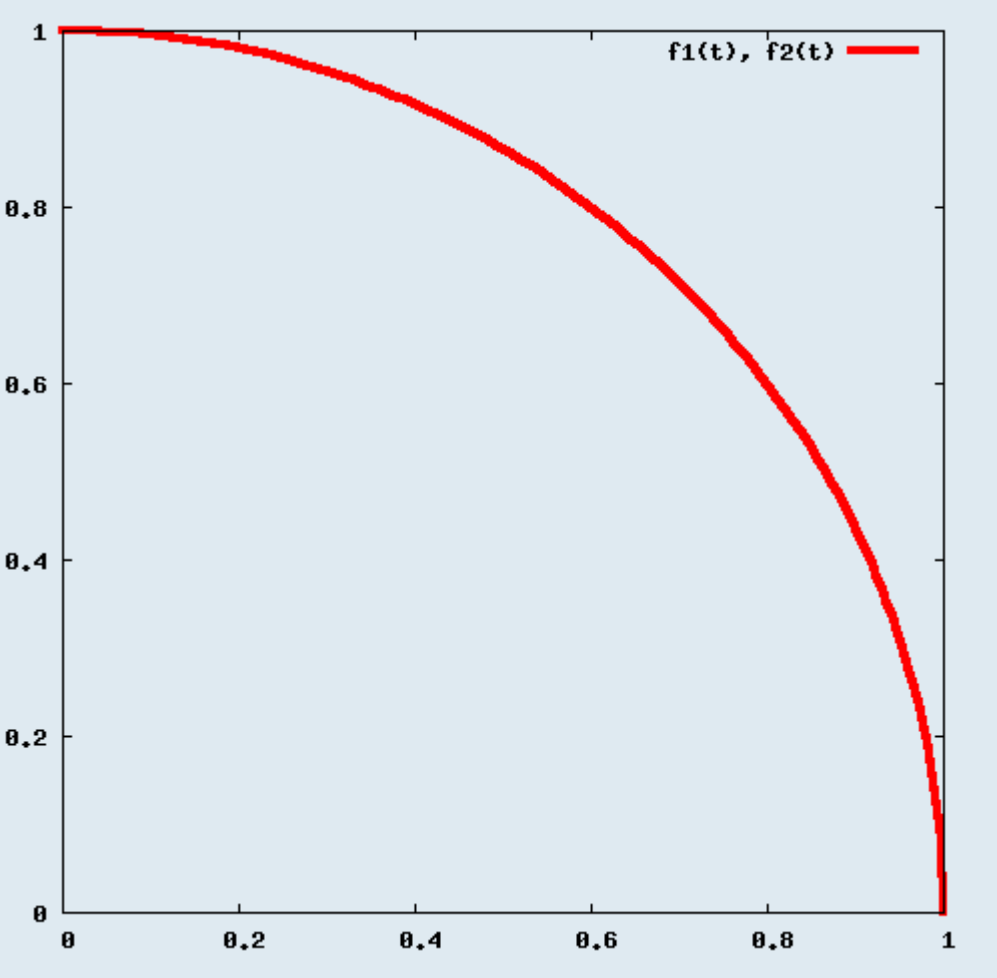

$$P(t) = (1-t)^3 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$+3 \cdot t \cdot (1-t)^2 \cdot \begin{pmatrix} x \\ 1 \end{pmatrix}$$
$$+3 \cdot t^2 \cdot (1-t) \cdot \begin{pmatrix} 1 \\ x \end{pmatrix}$$
$$+\mathrm{t}^3 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$P(0.5) = \begin{pmatrix} \cos(45°) \\ \sin(45°) \end{pmatrix}$$

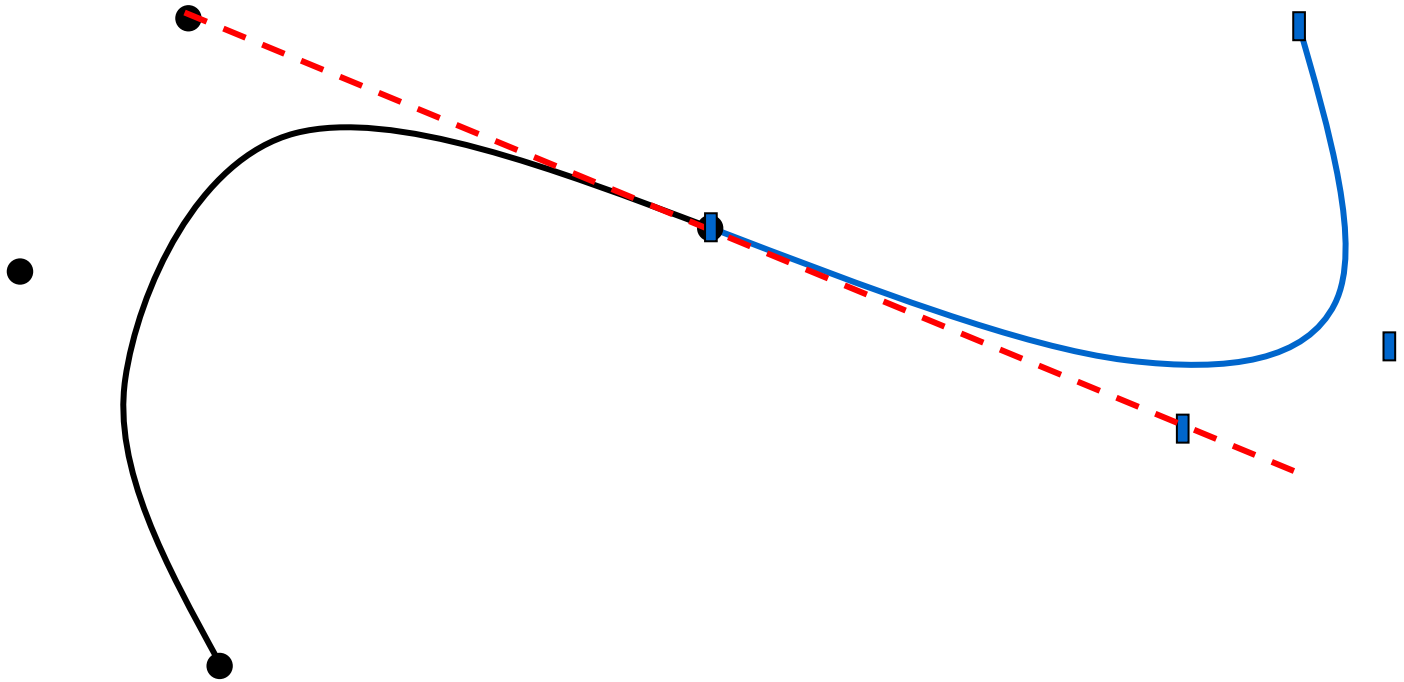
# Bezier-Viertelkreis

$$f_1(t) = 3 \cdot t \cdot (1-t)^2 \cdot 0.551897 + 3 \cdot t^2 \cdot (1-t) + t^3$$

$$f_2(t) = (1-t)^3 + 3 \cdot t \cdot (1-t)^2 + 3 \cdot t^2 \cdot (1-t) \cdot 0.551897$$



# Bézier-Kurven aneinandersetzen



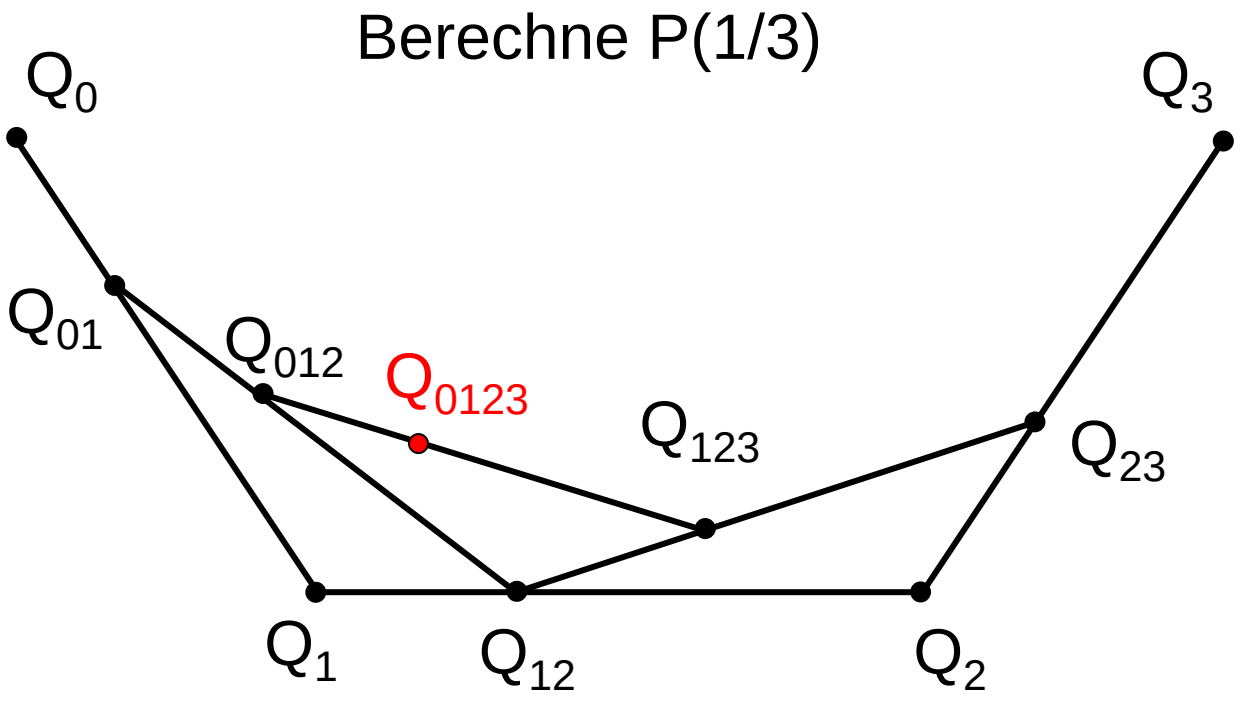
# Bézier-Kurve nach de Casteljau

$$P_{0,n}(t) = (1-t) \cdot P_{0,n-1}(t) + t \cdot P_{1,n}(t)$$

$$P_{0,3}(\tfrac{1}{3})$$

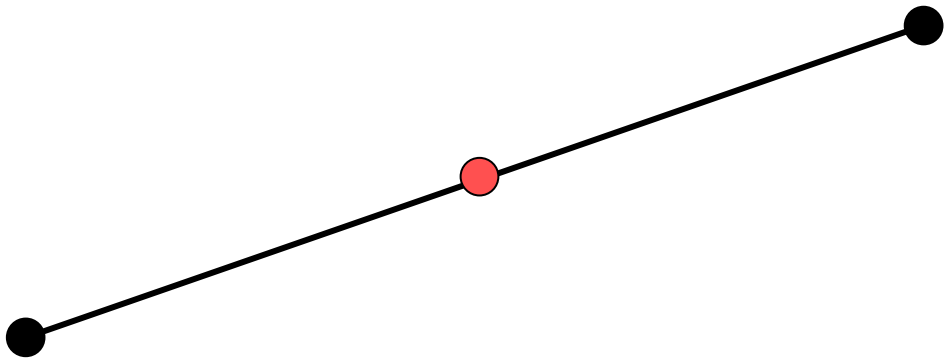
$$\overbrace{\underbrace{\tfrac{2}{3} \cdot P_{0,2}(\tfrac{1}{3})}_{\tfrac{2}{3} \cdot P_{0,1}(\frac{1}{3}) + \frac{1}{3} \cdot P_{1,2}(\frac{1}{3})} + \underbrace{\tfrac{1}{3} \cdot P_{1,3}(\tfrac{1}{3})}_{\tfrac{2}{3} \cdot P_{1,2}(\frac{1}{3}) + \frac{1}{3} \cdot P_{2,3}(\frac{1}{3})}}$$

# Berechnung nach de Casteljau



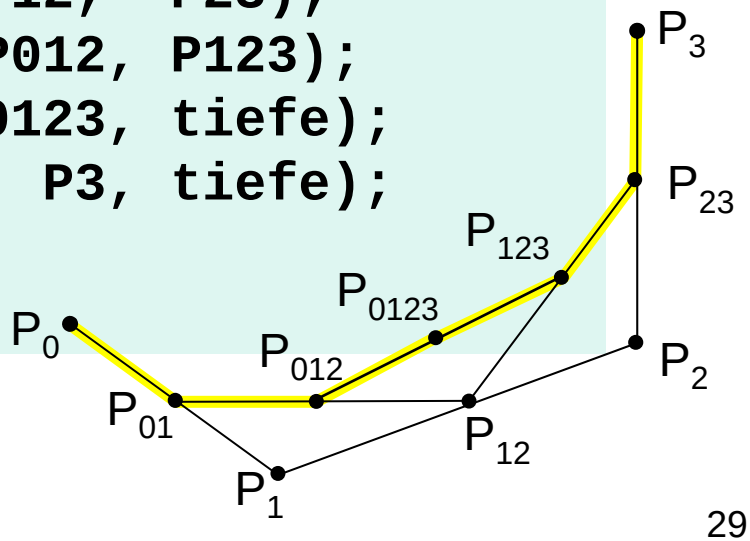
# halbiere

```
Point halbiere(Point P1, Point P2){
  Point P = new Point();
  P.x = (P1.x + P2.x)/2;
  P.y = (P1.y + P2.y)/2;
  return P;
}
```



# Rekursion nach de Casteljau

```
void bezier(Point P0, Point P1,
            Point P2, Point P3, int tiefe) {
if (tiefe == 0) drawLine(P0, P3); else {
    tiefe--;
    Point P01   = halbiere(P0,   P1);
    Point P12   = halbiere(P1,   P2);
    Point P23   = halbiere(P2,   P3);
    Point P012  = halbiere(P01,  P12);
    Point P123  = halbiere(P12,  P23);
    Point P0123 = halbiere(P012, P123);
    bezier(P0, P01, P012, P0123, tiefe);
    bezier(P0123, P123, P23, P3, tiefe);
  }
}
```



# Java-Applet zu Bézier-Kurven

~cg/2014/skript/Applets/Splines/App.html

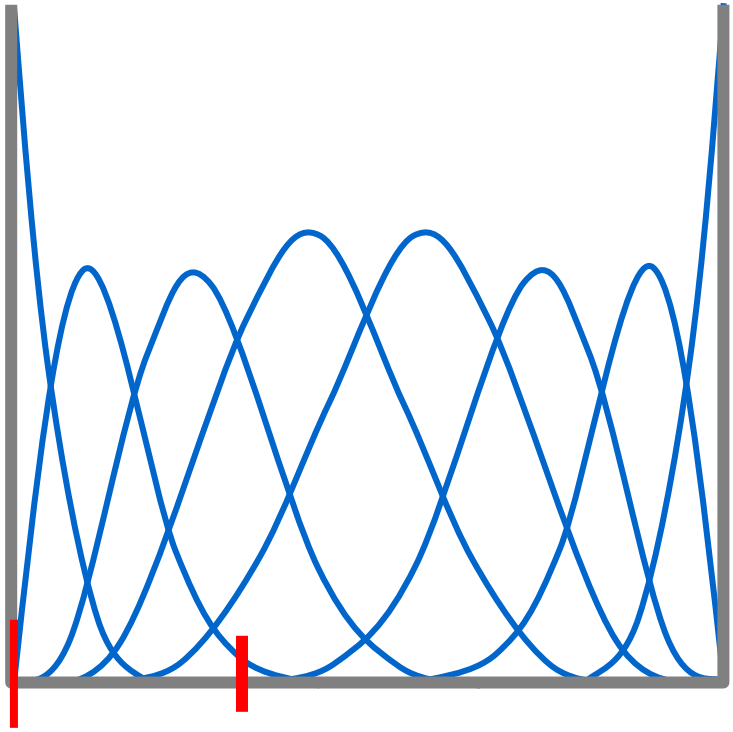
# Bewertung von Bézier-Kurven

- Grad der Polynome
  ist abhängig von Zahl der Kontrollpunkte
- Alle Kontrollpunkte
  wirken auf die gesamte Kurve

Wunsch:

- fester Polynomgrad
- lokaler Einfluss

# Wunsch: Lokaler Einfluss



z.B. Einfluss von maximal vier Kontrollpunkten

# B-Splines

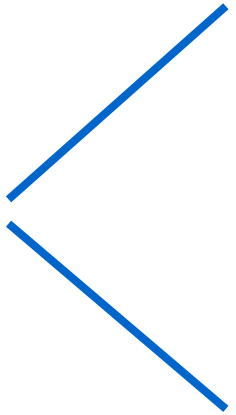
- wähle n+1 Kontrollpunkte $P_0$, $P_1$, …, $P_n$
- wähle k
- wähle Knotenvektor $t_0$, $t_1$, $t_2$, …, $t_{n+k}$
- konstruiere n+1 Gewichtsfunktionen
- Gewichtsfunktion wirkt auf max. k Abschnitte
- Gewichtsfunktion ist Polynom vom Grad k-1 (abschnittsweise)

# Konstruktion von B-Splines

$$T = (t_0, t_1, \ldots, t_{n+k}), t_j \leq t_{j+1}$$

$$N_{j,1}(t) = \begin{cases} 1 & \text{falls } t_j \leq t < t_{j+1} \\ 0 & \text{sonst} \end{cases} \qquad j = 0, \ldots, n+k-1$$

$$N_{j,i}(t) = \frac{t - t_j}{t_{j+i-1} - t_j} \cdot N_{j,i-1}(t) \; + \; \frac{t_{j+i} - t}{t_{j+i} - t_{j+1}} \cdot N_{j+1,i-1}(t)$$



$$P(t) = \sum_{i=0}^{n} N_{i,k}(t) \cdot P_i$$

# Beispiel für Knotenvektor

$$t_j = \begin{cases} 0 & \text{falls } j < k \\ j - k + 1 & \text{falls } k \leq j \leq n \\ n - k + 2 & \text{falls } j > n \end{cases}$$

$$j \in \{0, \ldots, n + k\} \qquad t \in [0, n - k + 2]$$

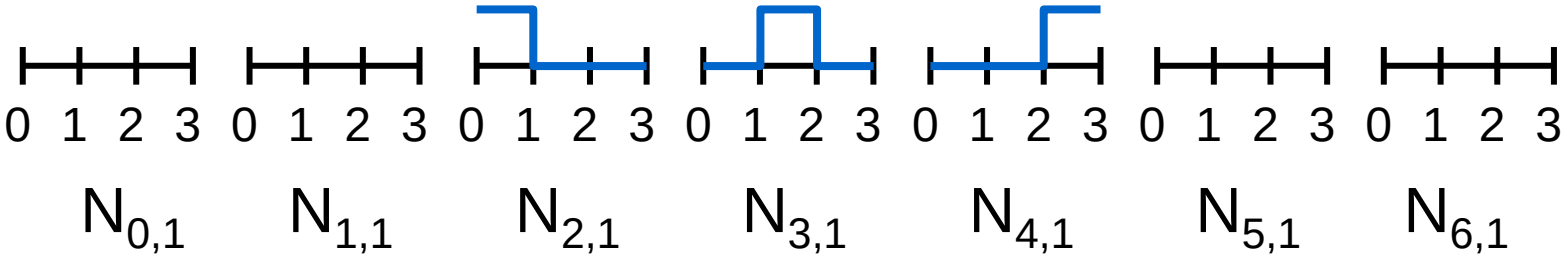
Beispiel: $n = 4,\ k = 3$

Knotenvektor: $T = (0, 0, 0, 1, 2, 3, 3, 3)$

Beispiel: $n = 8,\ k = 4$

Knotenvektor: $T = (0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 6, 6, 6)$

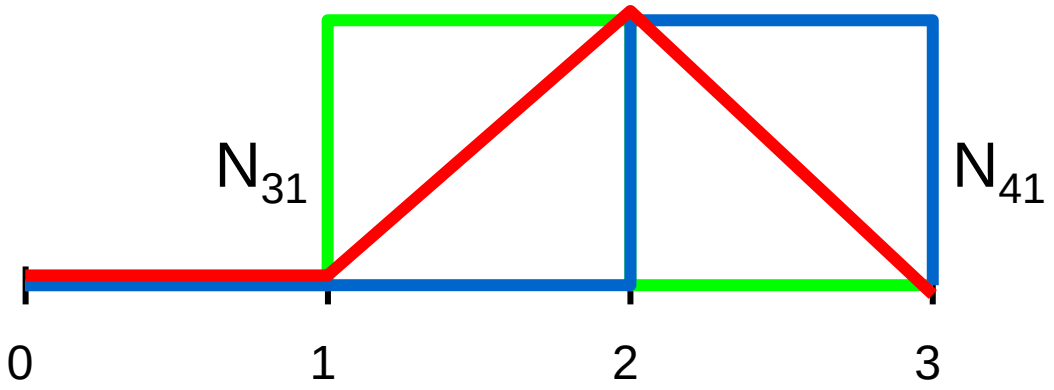
# Konstruktion von $N_{i,1}$ für n=4, k=3




n+k Ausgangsfunktionen

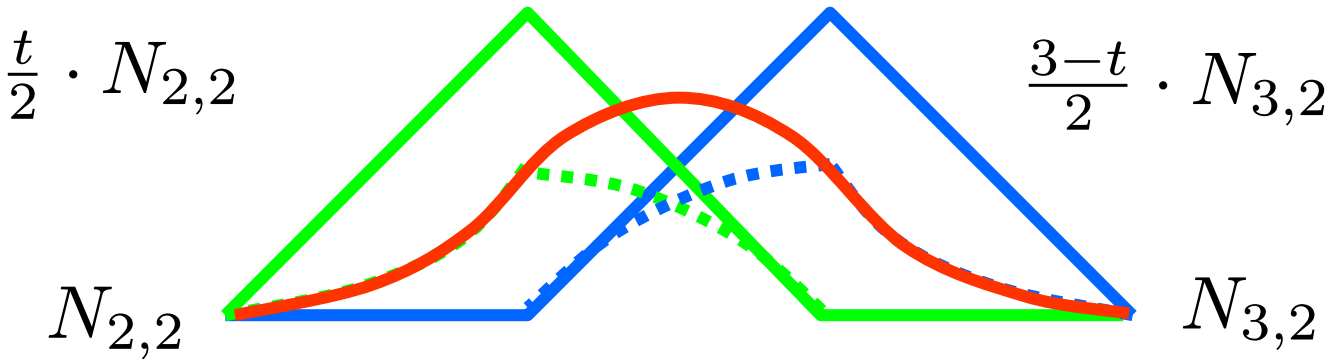
# Konstruktion von $N_{3,2}$

$$N_{3,2} := (t-1) \cdot N_{3,1} + (3-t) \cdot N_{4,1}$$



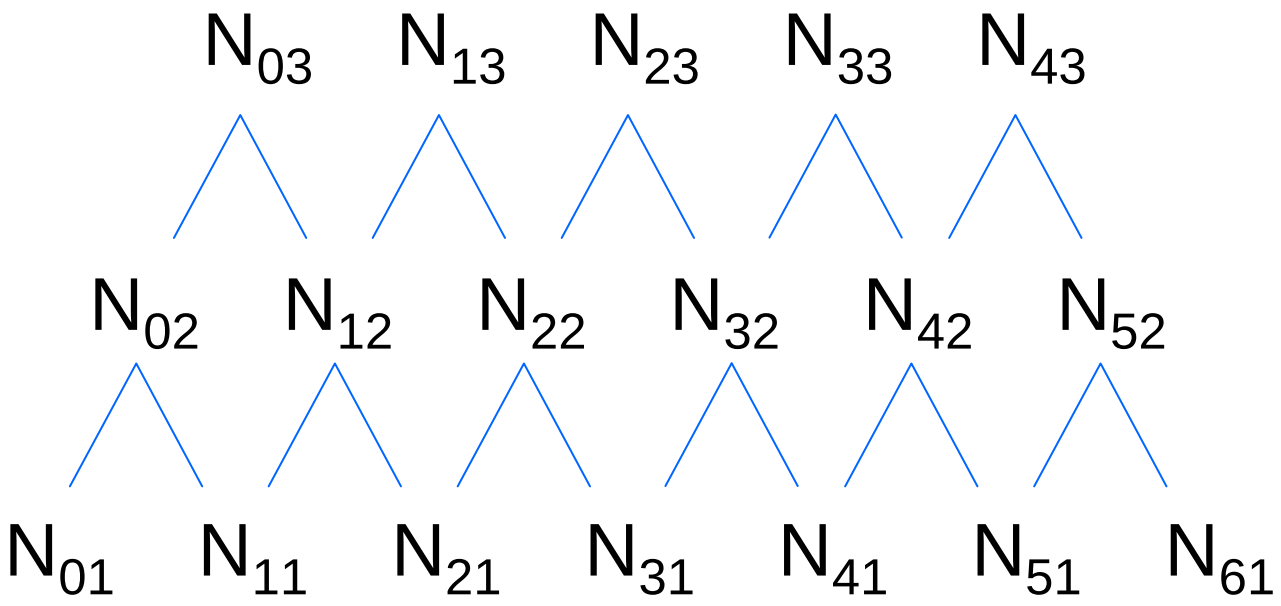
# Konstruktion von $N_{2,3}$

$$N_{2,3} = \frac{t}{2} \cdot N_{2,2} + \frac{3-t}{2} \cdot N_{3,2}$$



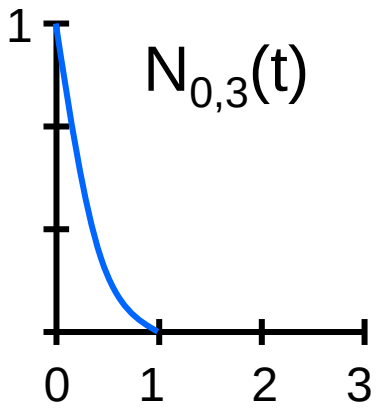
# Konstruktion von $N_{i,k}$

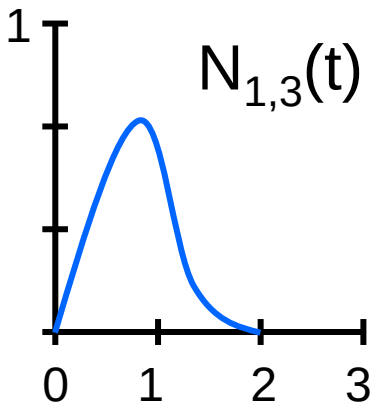
n + 1 Gewichtsfunktionen

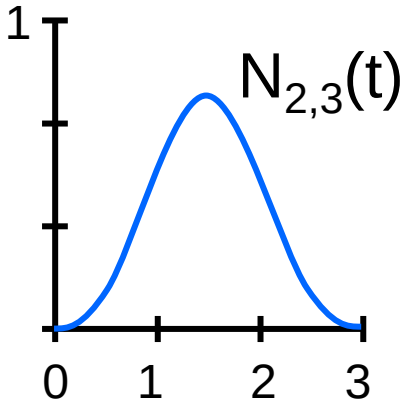


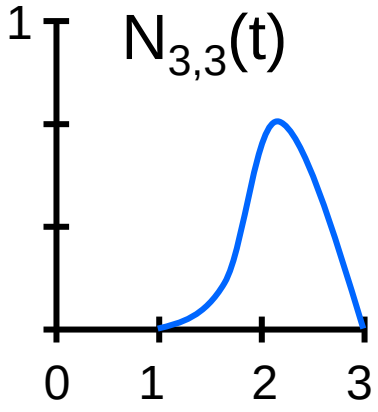

n + k Ausgangsfunktionen

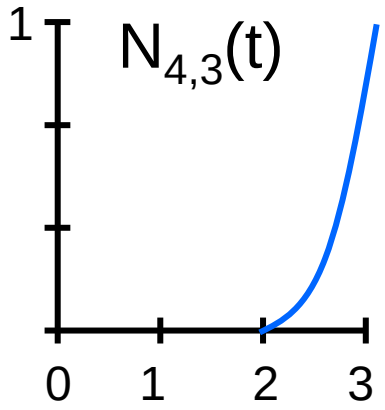
# Gewichtspolynome für n=4, k=3




$$P(t) = \sum_{i=0}^{4} N_{i,3}(t) \cdot P_i$$

# Sonderfall

$$k = n + 1$$

$$T = (\underbrace{0, ..., 0}_{k \text{ mal}}, \underbrace{1, ..., 1}_{k \text{ mal}}).$$

$$N_{i,k}(t) = \frac{(k-1)!}{i! \cdot (k-1-i)!} \cdot t^i \cdot (1-t)^{k-1-i}$$

$$= \begin{pmatrix} n \\ i \end{pmatrix} \cdot t^i \cdot (1-t)^{n-i} \quad = B_{i,n}(t)$$

$\Rightarrow$ Bernstein-Polynome sind Spezialfall von B-Splines

# Java-Applet von B-Splines

~cg/2014/skript/Applets/Splines/App.html

# Bewertung von B-Splines

- tolle Sache !

aber:

- kein Kreis
- nur invariant bzgl. affiner Abbildungen

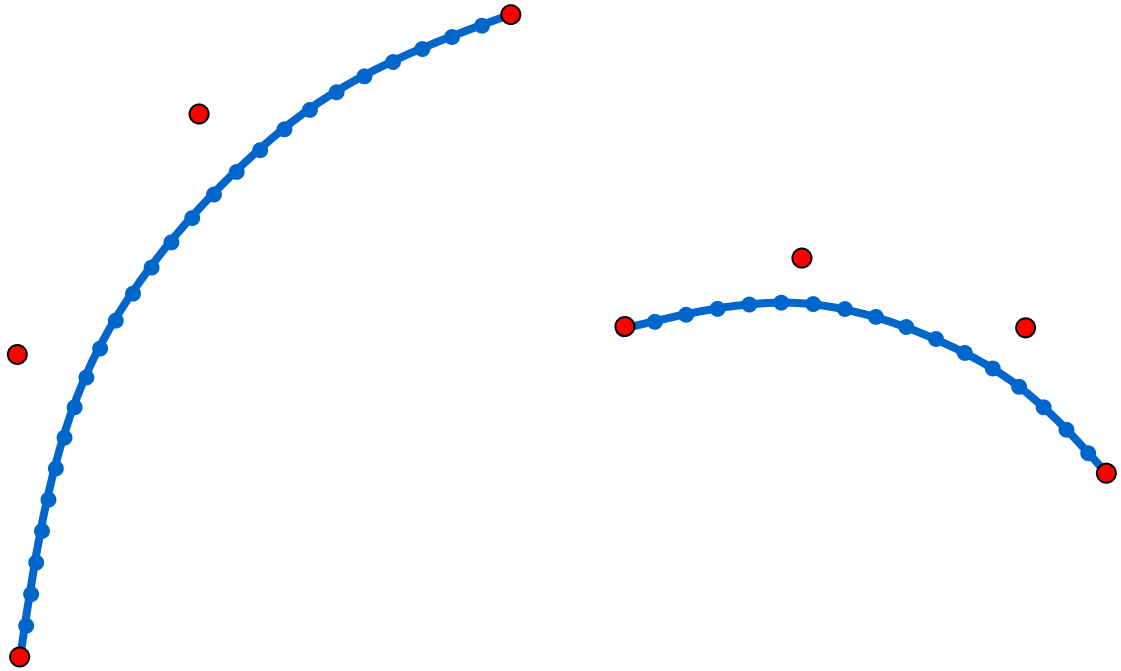
# Affine Abbildung

$$\vec{y} = A \cdot \vec{x} + \vec{b}$$

- Translation
- Skalierung
- Rotation
- Scherung
- Spiegelung

sind affine Abbildungen

# Invarianz bzgl Abbildung



B-Splines sind nicht invariant unter Projektion !

# NURBS

| | |
|---|---|
| Non Uniform | $t_j$ nicht äquidistant |
| Rational | Gewichtsquotient |
| B-Splines | Gewichtsfunktion |

# NURBS

Punkte $P_i$ Knoten $t_j$ Gewichte $h_i$

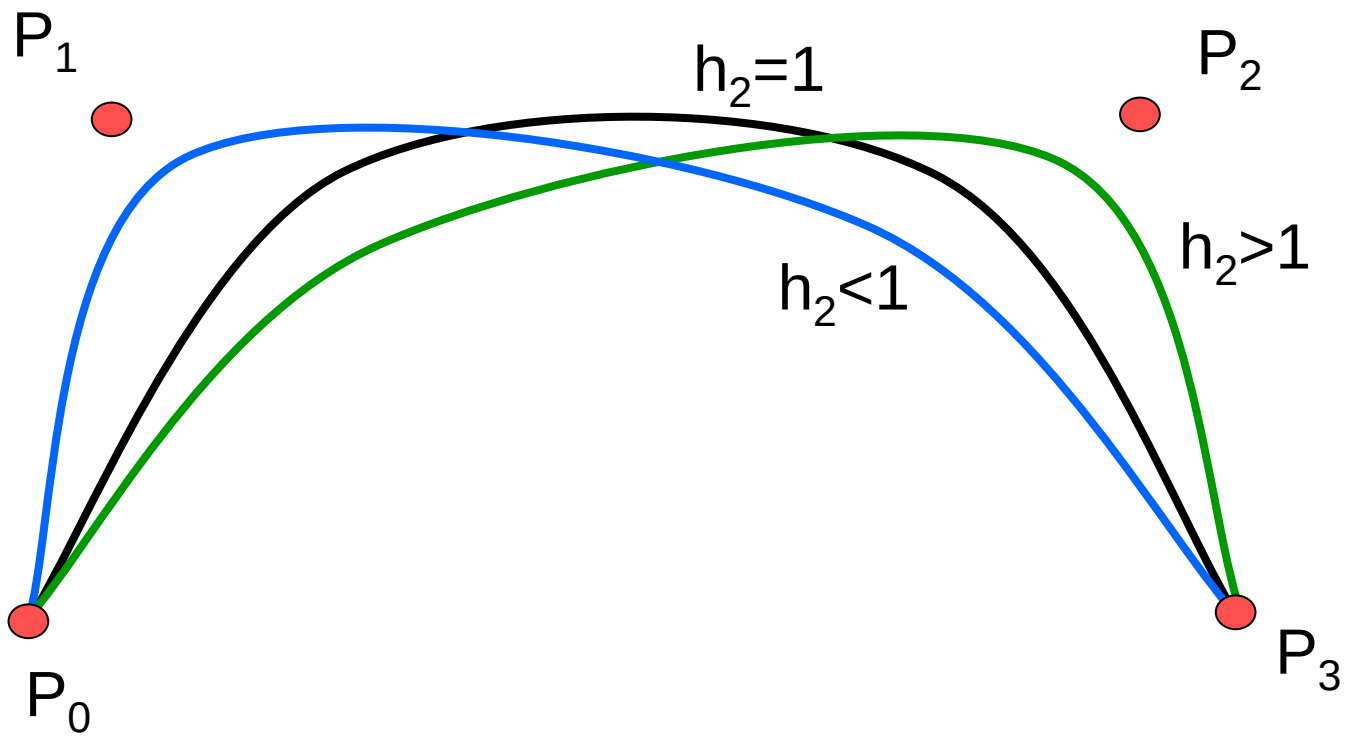
$$\sum_{i=0}^{n} N_{i,k}(t) = 1 \qquad \sum_{i=0}^{n} h_i \cdot N_{i,k}(t) = z$$

normiere auf 1

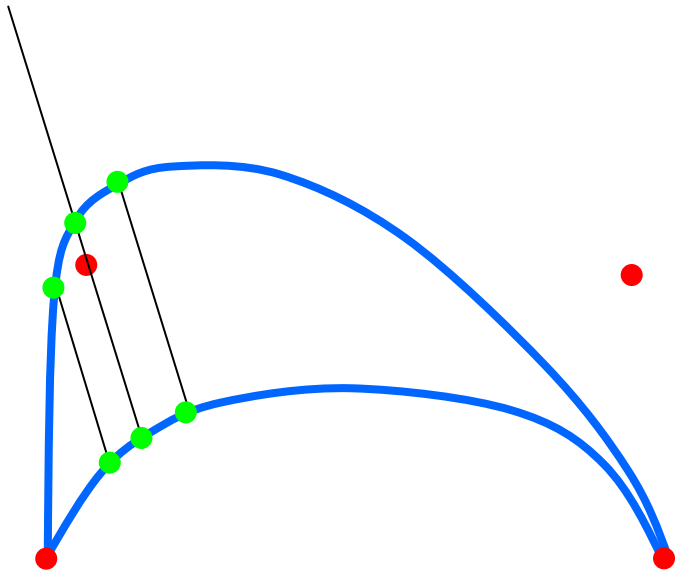
$$R_{i,k}(t) = \frac{h_i \cdot N_{i,k}(t)}{\sum_{j=0}^{n} h_j \cdot N_{j,k}(t)}$$

$$P(t) = \sum_{i=0}^{n} R_{i,k}(t) \cdot P_i$$

# Auswirkung der NURBS-Gewichte

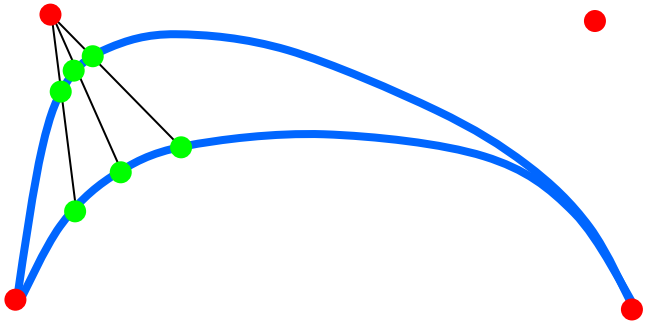


# Möglichkeiten der Einflussnahme



Kontrollpunkt verschieben:

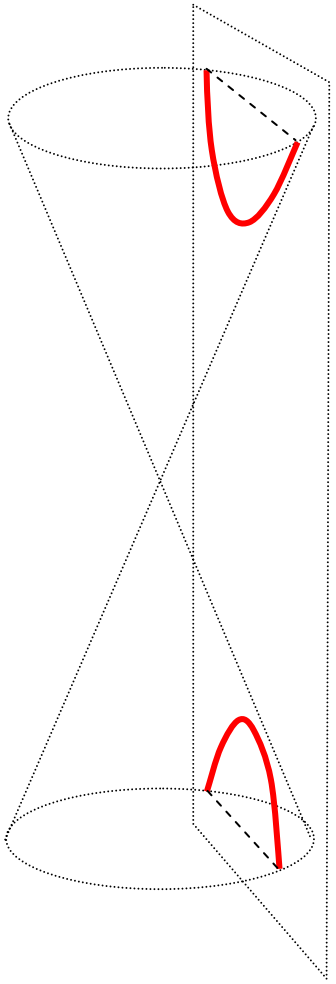
Punkte wandern parallel



Gewicht erhöhen:

Punkte wandern auf Kontrollpunkt zu

# Kegelschnitte



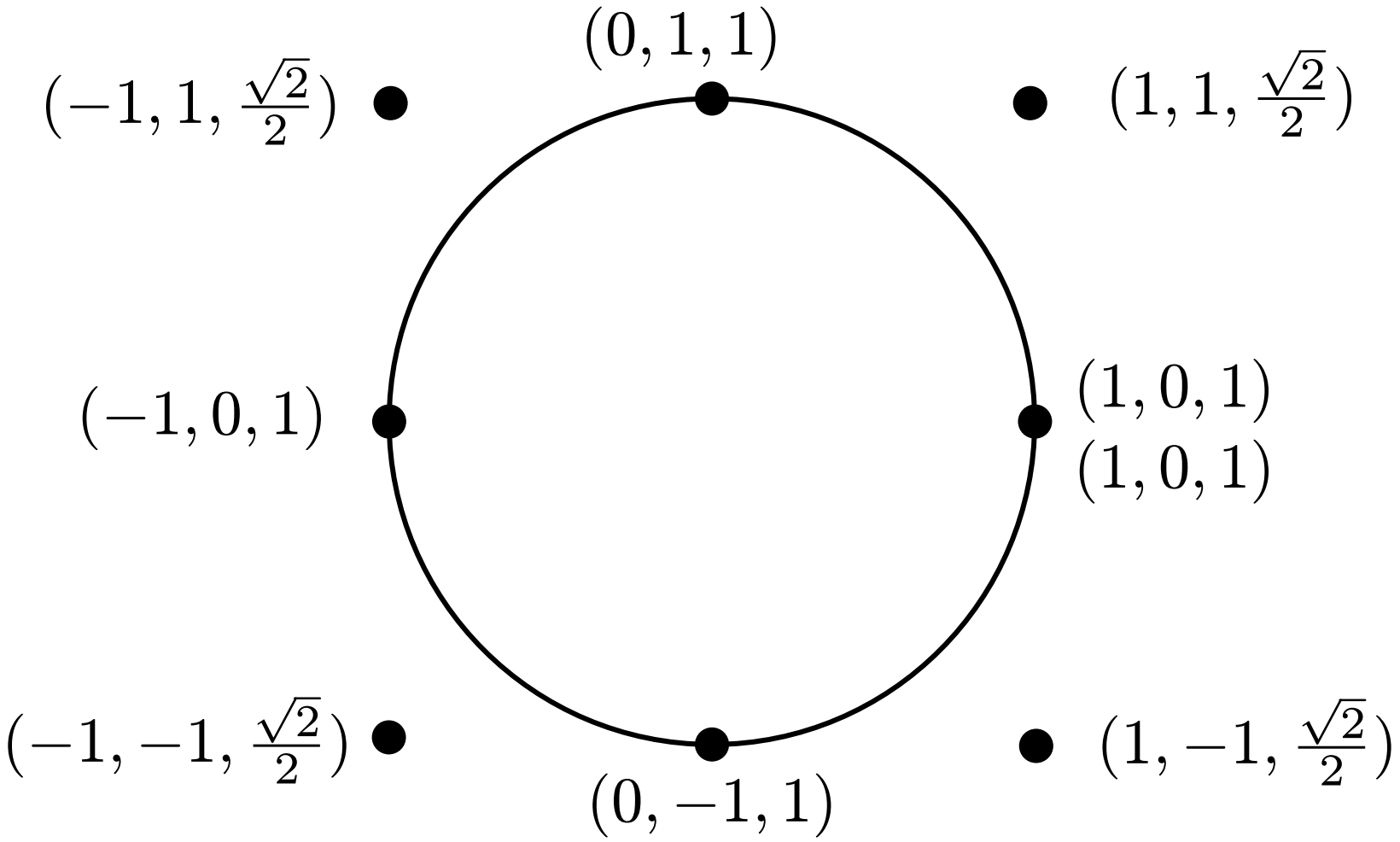
**Hyperbel**
Differenz der Abstände konstant

**Parabel**
Abstand zu Punkt und Gerade gleich

**Kreis**
Abstand zu Punkt konstant

**Ellipse**
Summe der Abstände konstant

# NURBS-Kreis




Knotenvektor: 0,0,0,1,1,2,2,3,3,4,4,4

# Viertelkreis durch NURBS

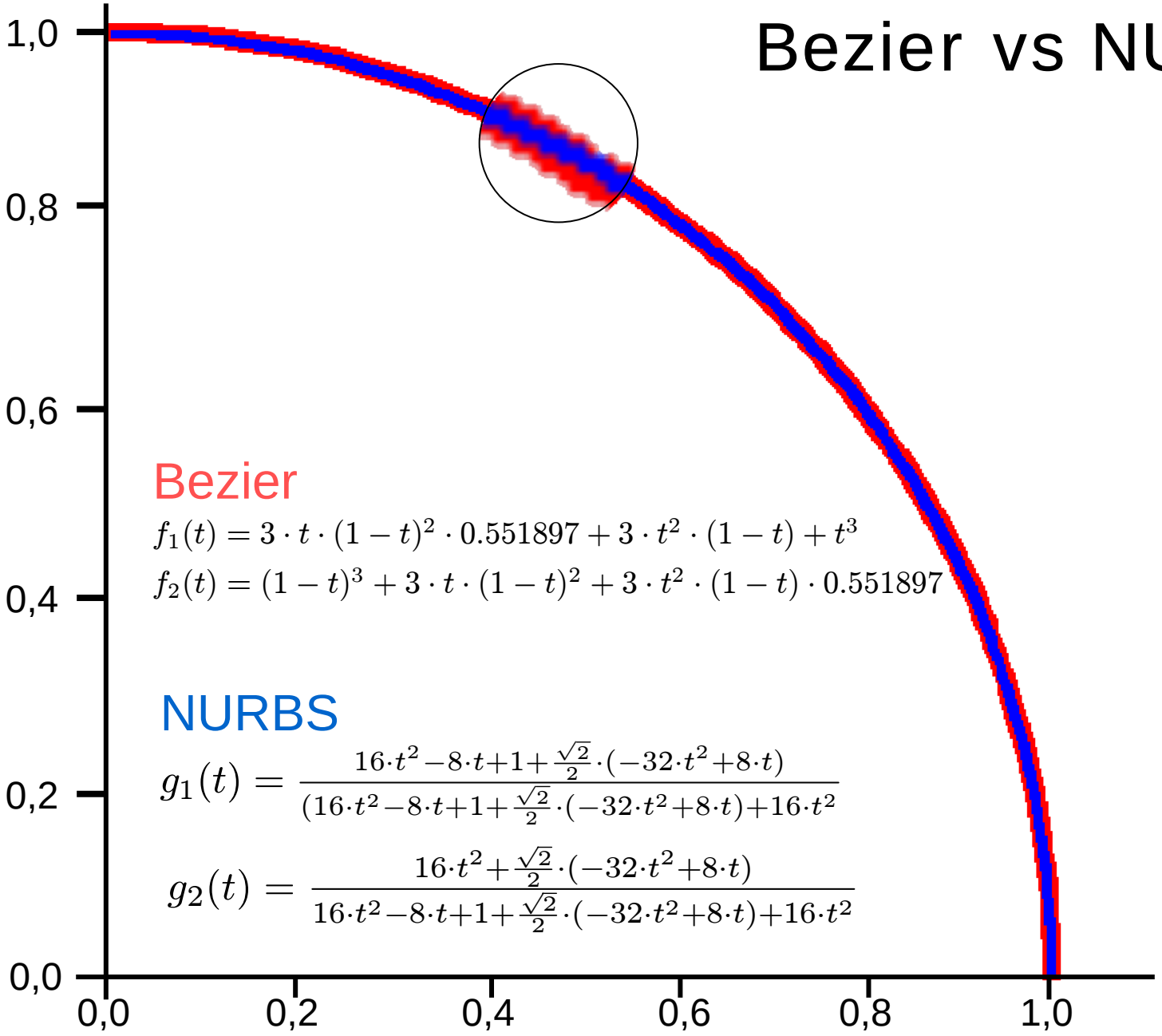
$$P_x(t) = \left( \frac{16t^2-8t+1+\frac{\sqrt{2}}{2}(-32t^2+8t)}{16t^2-8t+1+\frac{\sqrt{2}}{2}(-32t^2+8t)+16t^2} \right)$$

$$P_y(t) = \left( \frac{\frac{\sqrt{2}}{2}(-32t^2+8t)+16t^2}{16t^2-8t+1+\frac{\sqrt{2}}{2}(-32t^2+8t)+16t^2} \right)$$

korrekter Kreis, denn $P_x^2(t) + P_y^2(t) =$

$$\left( \frac{16t^2-8t+1+\frac{\sqrt{2}}{2}(-32t^2+8t)}{16t^2-8t+1+\frac{\sqrt{2}}{2}(-32t^2+8t)+16t^2} \right)^2 + \left( \frac{\frac{\sqrt{2}}{2}(-32t^2+8t)+16t^2}{16t^2-8t+1+\frac{\sqrt{2}}{2}(-32t^2+8t)+16t^2} \right)^2 = \quad 1$$

# Bezier vs NURBS




## Bezier

$$f_1(t) = 3 \cdot t \cdot (1-t)^2 \cdot 0.551897 + 3 \cdot t^2 \cdot (1-t) + t^3$$

$$f_2(t) = (1-t)^3 + 3 \cdot t \cdot (1-t)^2 + 3 \cdot t^2 \cdot (1-t) \cdot 0.551897$$

## NURBS

$$g_1(t) = \frac{16 \cdot t^2 - 8 \cdot t + 1 + \frac{\sqrt{2}}{2} \cdot (-32 \cdot t^2 + 8 \cdot t)}{(16 \cdot t^2 - 8 \cdot t + 1 + \frac{\sqrt{2}}{2} \cdot (-32 \cdot t^2 + 8 \cdot t) + 16 \cdot t^2}$$

$$g_2(t) = \frac{16 \cdot t^2 + \frac{\sqrt{2}}{2} \cdot (-32 \cdot t^2 + 8 \cdot t)}{16 \cdot t^2 - 8 \cdot t + 1 + \frac{\sqrt{2}}{2} \cdot (-32 \cdot t^2 + 8 \cdot t) + 16 \cdot t^2}$$


# Wer kurvt am besten ?

| | Splines | Bézier | B-Splines | NURBS |
|---|---|---|---|---|
| konstanter Polynomgrad | + | - | + | + |
| lokaler Einfluss | - | - | + | + |
| effiziente Speicherung | + | + | + | + |
| Kreis möglich | - | - | - | + |
| invariant bzgl. Affine Abb. | + | + | + | + |
| invariant bzgl. Projektion | - | - | - | + |