Computergrafik SS 2016

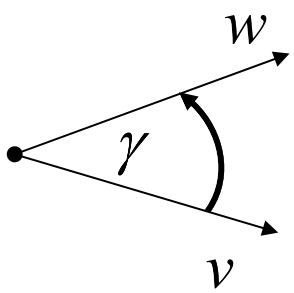
Oliver Vornberger

Vorlesung vom 10.05.2016

noch Kapitel 12:
Mathematische Grundlagen

Skalarprodukt

$$\vec{v} \cdot \vec{w} := \sum_{i=1}^{n} v_i \cdot w_i$$



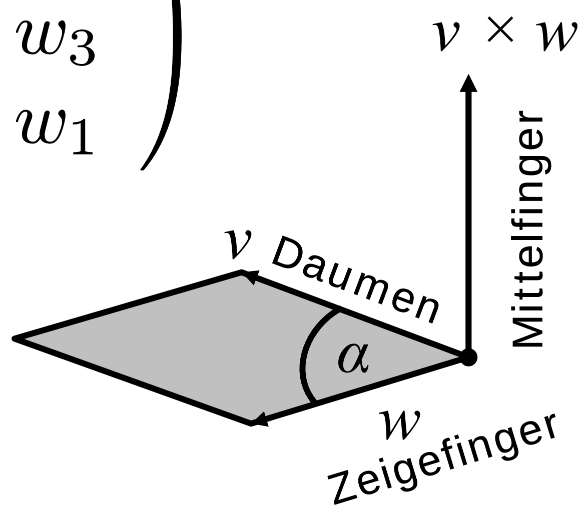
$$\cos(\gamma) = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}| \cdot |\vec{w}|}$$

Kreuzprodukt

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$\vec{v} \times \vec{w} = \begin{pmatrix} v_2 \cdot w_3 - v_3 \cdot w_2 \\ v_3 \cdot w_1 - v_1 \cdot w_3 \\ v_1 \cdot w_2 - v_2 \cdot w_1 \end{pmatrix}$$

$$|\vec{v} \times \vec{w}| = |\vec{v}| \cdot |\vec{w}| \cdot \sin(\alpha)$$



Adjunkte A_{ij}

A_{ij} := $(-1)^{i+j}$ · Unterdeterminante bzgl i,j

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & \boxed{a_{23}} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

Berechnung der Matrix-Inversion

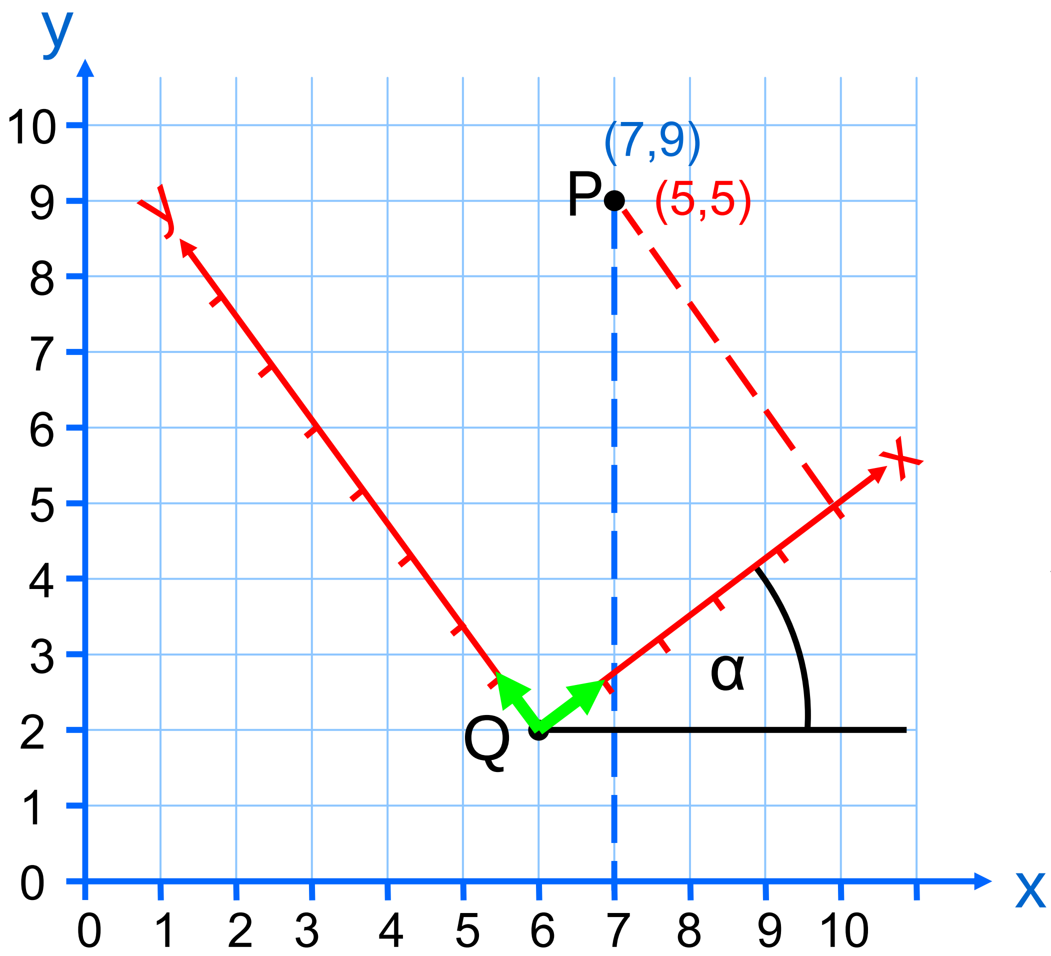
Wähle Zeile i

$$\det(\mathrm{A}) = \sum_{k=1}^{4} a_{ik} A_{ik}$$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} A_{11} & A_{21} & A_{31} & A_{41} \\ A_{12} & A_{22} & A_{32} & A_{42} \\ A_{13} & A_{23} & A_{33} & A_{43} \\ A_{14} & A_{24} & A_{34} & A_{44} \end{pmatrix}$$

Koordinatensystemwechsel, die 2.

beschreibe P aus Sicht von blau



drehe rotes System
um α nach rechts ⇒

drehe P um α nach links

verschiebe um Q

$\cos(\alpha) = 4/5 = 0.8$

$\sin(\alpha) = 3/5 = 0.6$

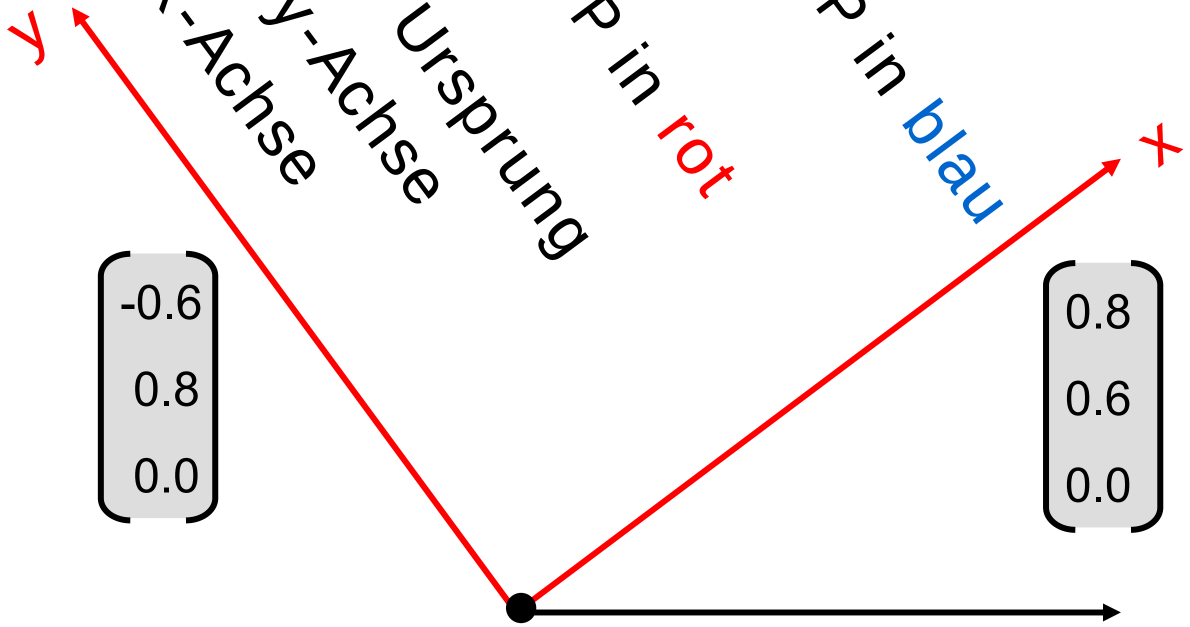
$Q = (6,2)$

$$\begin{pmatrix} 0.8 & -0.6 & 6.0 \\ 0.6 & 0.8 & 2.0 \\ 0.0 & 0.0 & 1.0 \end{pmatrix}$$

Koordinatensystemwechsel, die 3.

$$M_{B\to A} \begin{pmatrix} 0.8 & -0.6 & 6.0 \\ 0.6 & 0.8 & 2.0 \\ 0.0 & 0.0 & 1.0 \end{pmatrix} \cdot \begin{pmatrix} 5.0 \\ 5.0 \\ 1.0 \end{pmatrix} = \begin{pmatrix} 7.0 \\ 9.0 \\ 1.0 \end{pmatrix}$$

x-Achse y-Achse Ursprung P in rot P in blau



Koordinatensystemwechsel, die 5.

Gegeben Punkt P_A, beschrieben in A

Gesucht Punkt P_B, beschrieben in B

$$M_{\mathcal{B}\rightarrow\mathcal{A}} \cdot \vec{p}_{\mathcal{B}} = \vec{p}_{\mathcal{A}}$$

$$\vec{p}_{\mathcal{B}} = M_{\mathcal{B}\rightarrow\mathcal{A}}^{-1} \cdot \vec{p}_{\mathcal{A}}$$

Multipliziere P_A mit der Inversen von $M_{B\rightarrow A}$

Computergrafik SS 2016

Oliver Vornberger

Kapitel 13:
3D-Transformationen

Einsatzgebiet

- Platzierung von Objekten in der Szene
- Berechnung der Projektion

Translation

$$(x', y', z') := (x + t_x, y + t_y, z + t_z)$$

$$T(t_x, t_y, t_z) = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Skalierung

Fixpunkt im Ursprung:

$$(x', y', z') := (x \cdot s_x, y \cdot s_y, z \cdot s_z)$$

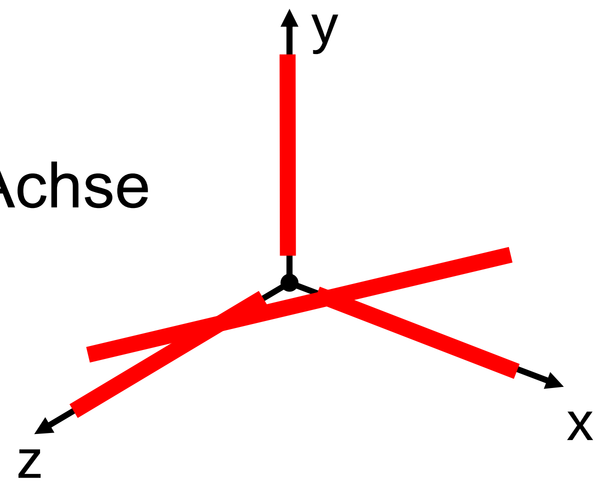
$$S(s_x, s_y, s_z) = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Fixpunkt bei Z_x, Z_y, Z_z:

$$T(Z_x, Z_y, Z_z) \cdot S(s_x, s_y, s_z) \cdot T(-Z_x, -Z_y, -Z_z)$$

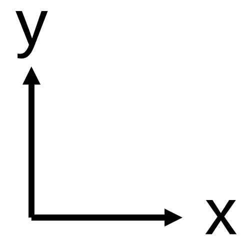
Rotation

- um z-Achse
- um x-Achse
- um y-Achse
- um beliebige Achse



Rotation um z-Achse

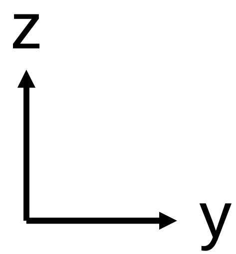
$$
\begin{aligned}
x' &:= x \cdot \cos(\delta) - y \cdot \sin(\delta) \\
y' &:= x \cdot \sin(\delta) + y \cdot \cos(\delta) \\
z' &:= z
\end{aligned}
$$



$$
R_z(\delta) = \begin{pmatrix} \cos(\delta) & -\sin(\delta) & 0 & 0 \\ \sin(\delta) & \cos(\delta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Rotation um x-Achse

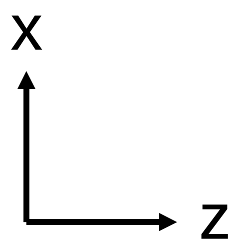
$$\begin{aligned} x' &:= x \\ y' &:= y \cdot \cos(\delta) - z \cdot \sin(\delta) \\ z' &:= y \cdot \sin(\delta) + z \cdot \cos(\delta) \end{aligned}$$



$$R_x(\delta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\delta) & -\sin(\delta) & 0 \\ 0 & \sin(\delta) & \cos(\delta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation um y-Achse

$$
\begin{aligned}
x' &:= z \cdot \sin(\delta) + x \cdot \cos(\delta) \\
y' &:= y \\
z' &:= z \cdot \cos(\delta) - x \cdot \sin(\delta)
\end{aligned}
$$

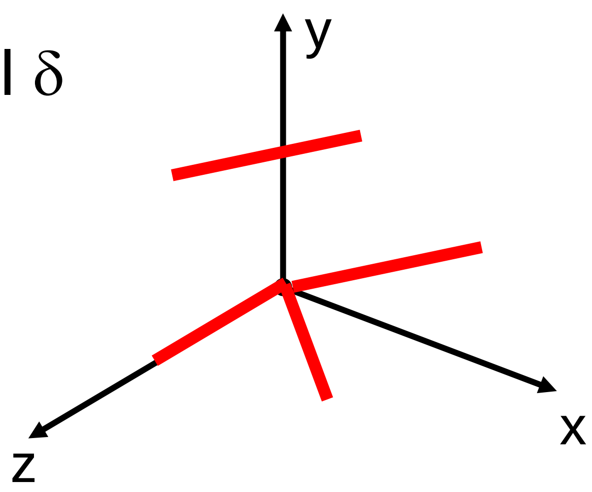


$$
R_y(\delta) = \begin{pmatrix} \cos(\delta) & 0 & \sin(\delta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\delta) & 0 & \cos(\delta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

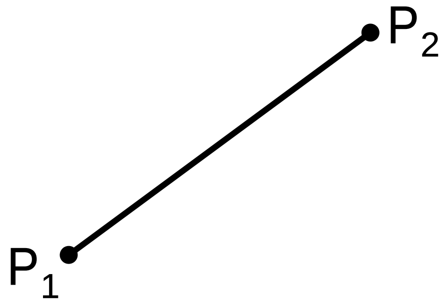
Rotation um beliebige Achse

Punkt P Drehwinkel δ Drehachse P_2-P_1

1. Translation in den Ursprung
2. Rotation um die x-Achse in die xz-Ebene
3. Rotation um die y-Achse in die z-Achse
4. Rotation um die z-Achse mit Winkel δ
5. Inversion von Schritt 3
6. Inversion von Schritt 2
7. Inversion von Schritt 1



Drehachse



$$\vec{v} = P_2 - P_1 = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

$$|\vec{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

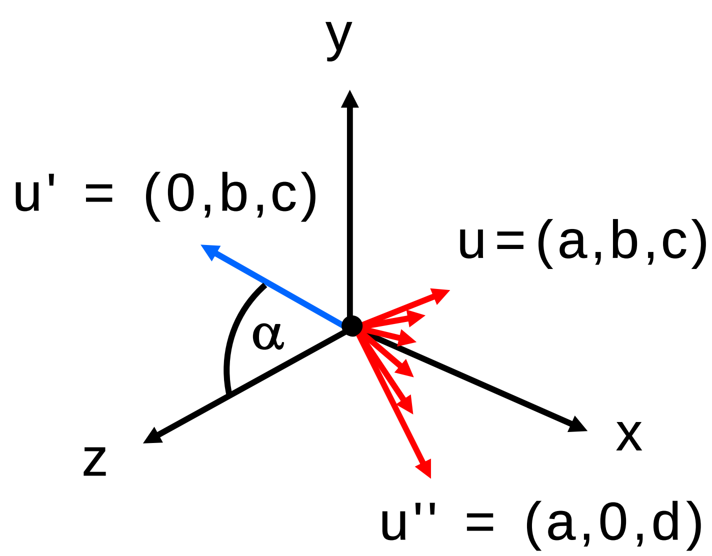
$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad |\vec{u}| = 1$$

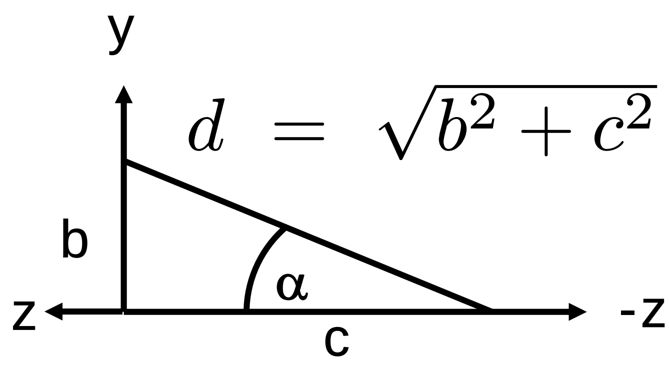
$$a = \frac{x_2 - x_1}{|\vec{v}|} \quad b = \frac{y_2 - y_1}{|\vec{v}|} \quad c = \frac{z_2 - z_1}{|\vec{v}|}$$

1.) Translation in den Ursprung

$$T(-x_1, -y_1, -z_1)$$

2.) Rotation um x-Achse in xz-Ebene





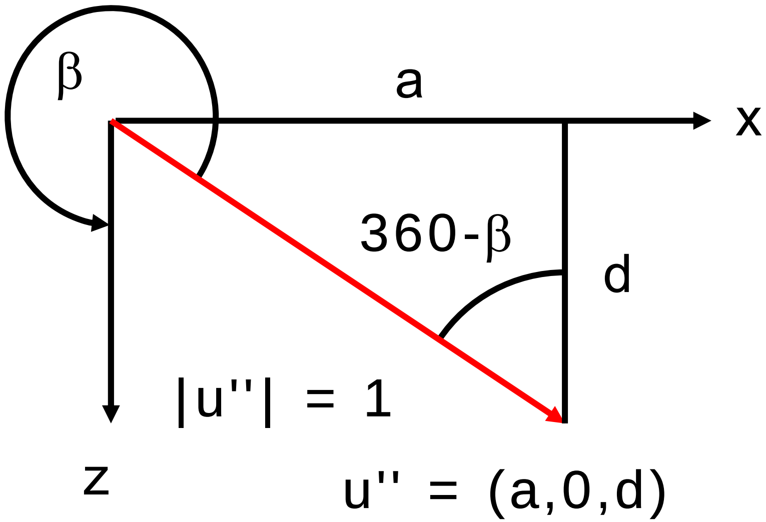
$$d = \sqrt{b^2 + c^2}$$

cos(α) = c/d

sin(α) = b/d

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3.) Rotation um y-Achse in z-Achse



$$R_y(\beta) = \begin{pmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \cos(\beta) = \quad \cos(360^\circ - \beta) = \quad d$$

$$\Rightarrow \sin(\beta) = -\sin(360^\circ - \beta) = -a$$

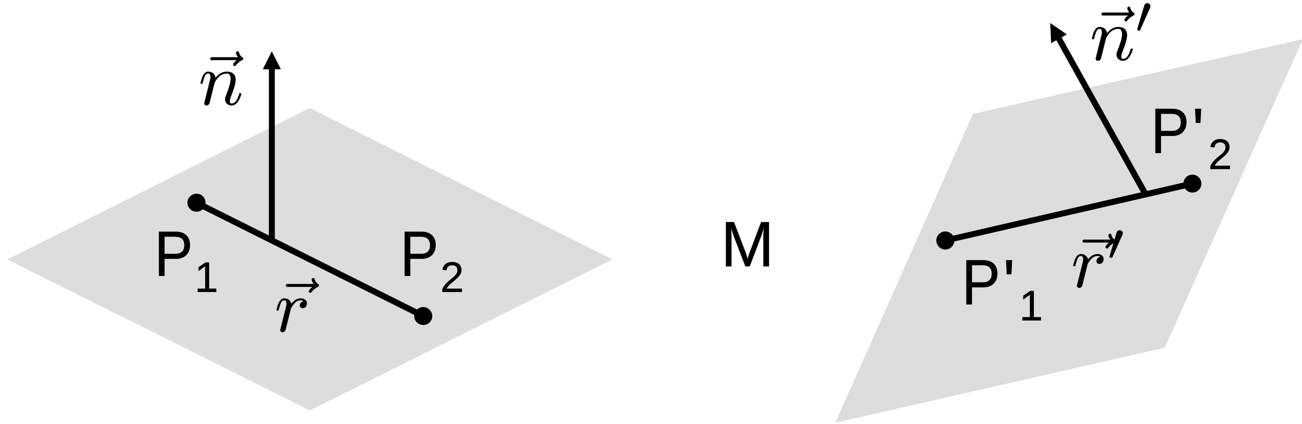
4.) Rotation um die z-Achse

$$R_z(\delta) = \begin{pmatrix} \cos(\delta) & -\sin(\delta) & 0 & 0 \\ \sin(\delta) & \cos(\delta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1. - 7.) Gesamttransformation

$$
\begin{aligned}
R(\vec{v}, \delta) \; = \; & T(P_1) \cdot R_x^{-1}(\alpha) \cdot R_y^{-1}(\beta) \cdot R_z(\delta) \cdot R_y(\beta) \cdot R_x(\alpha) \cdot T(-P_1)
\end{aligned}
$$

Transformation einer Normalen



$$\vec{r} = P_2 - P_1 \qquad P_1' = M \cdot P_1 \quad P_2' = M \cdot P_2$$

$$\vec{r}' = P_2' - P_1'$$

$$= M \cdot P_2 - M \cdot P_1 = M \cdot (P_2 - P_1) = M \cdot \vec{r}$$

$$\vec{n}^T \cdot \vec{r} = 0 \qquad M \cdot \vec{r} = \vec{r}' \qquad \vec{n}'^T \cdot \vec{r}' \overset{!}{=} 0$$

Umformungen

$$\vec{n}^T \cdot \vec{r} = 0$$

$$\vec{n}^T \cdot M^{-1} \cdot M \cdot \vec{r} = 0$$

$$((M^{-1})^T \cdot \vec{n})^T \cdot \underbrace{M \cdot \vec{r}}_{\vec{r}'} = 0$$

$$\vec{n}'^T \cdot \vec{r}' = 0$$

$$((M^{-1})^T \cdot \vec{n})^T = \vec{n}'^T$$

$$((M^{-1})^T \cdot \vec{n}) = \vec{n}'$$

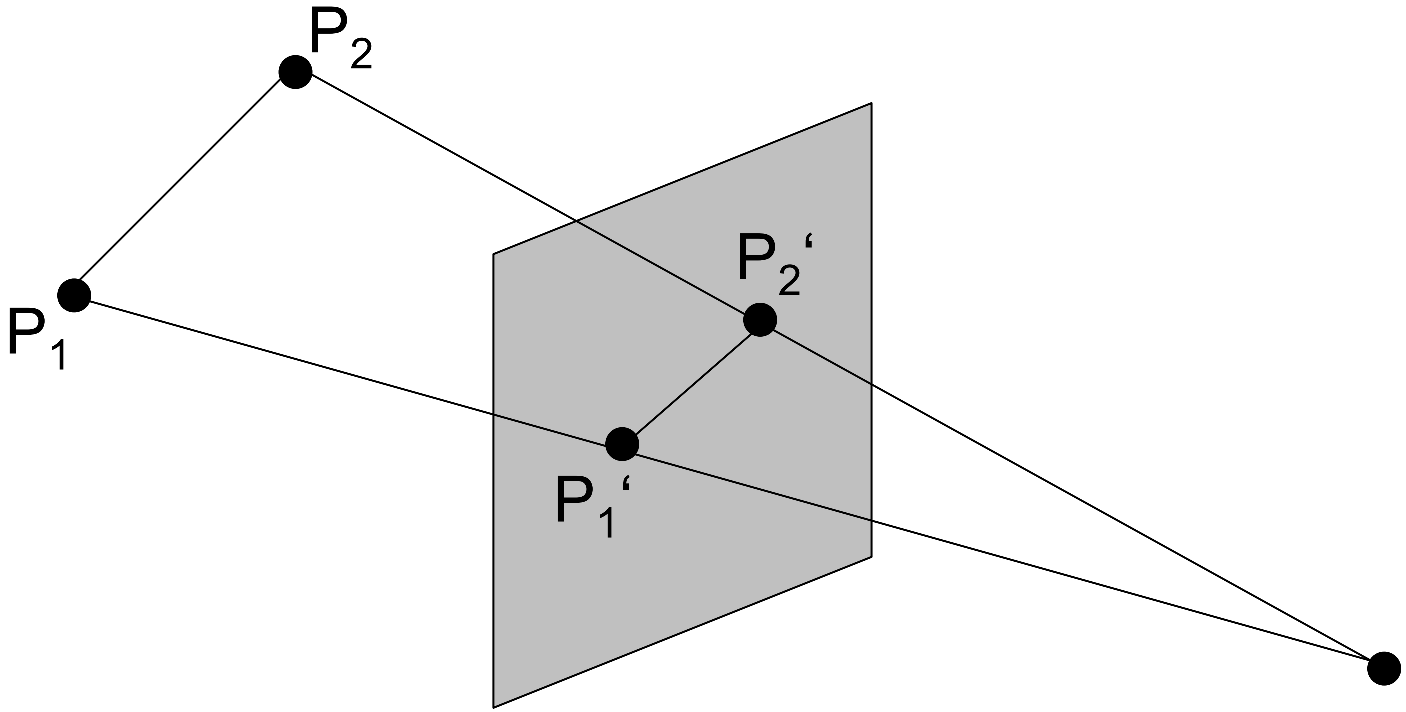
$\Rightarrow$ transformiere den Normalenvektor mit der transponierten Inversen !

Computergrafik SS 2016

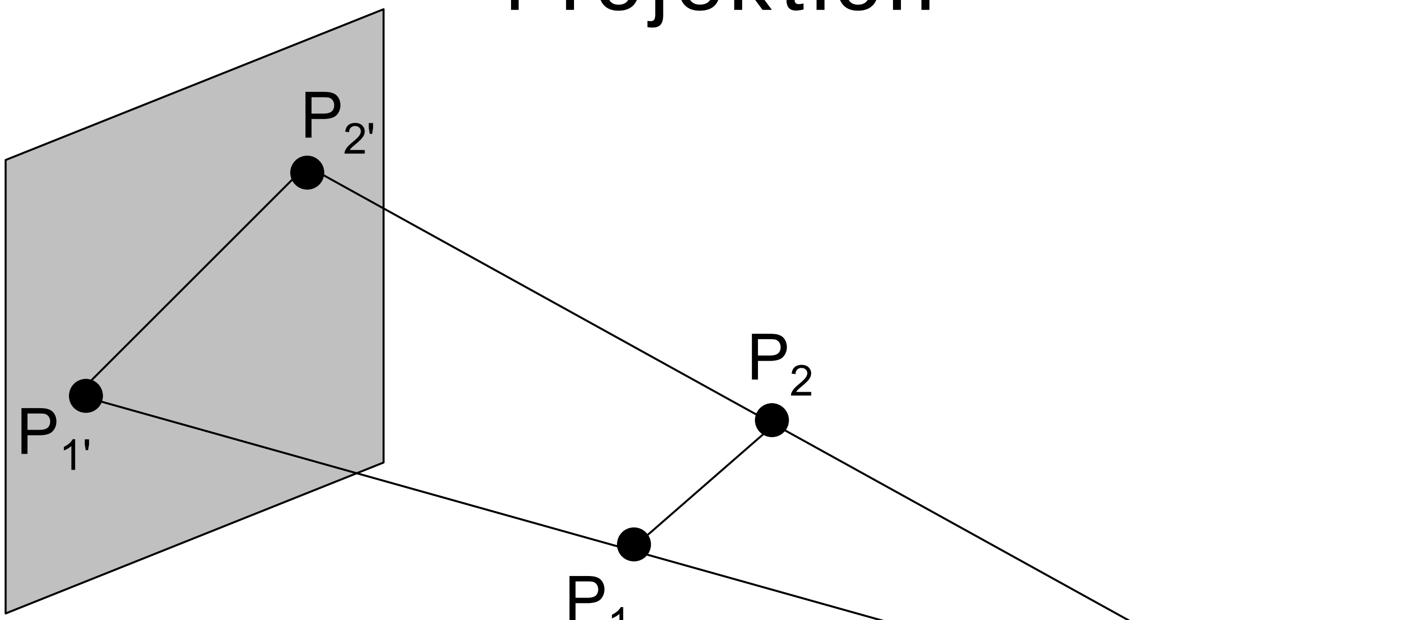
Oliver Vornberger

Kapitel 14:
Projektion

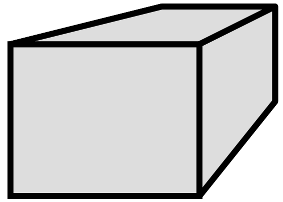
Projektion



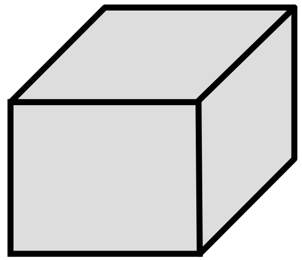
Projektion



Projektionsarten

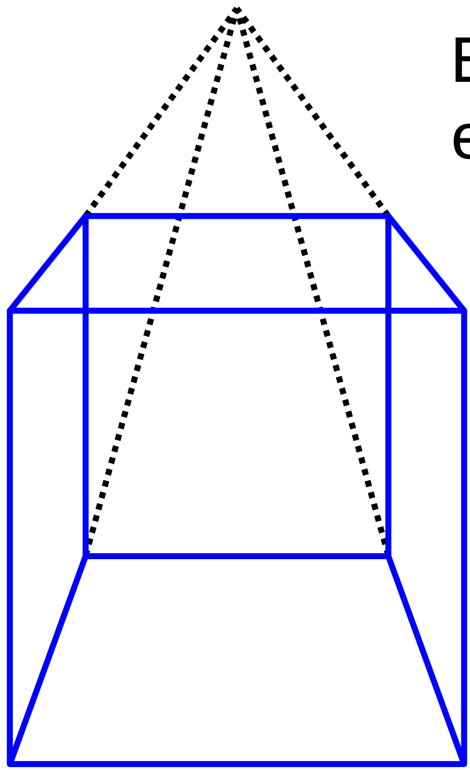


- Zentralprojektion:
  Augenpunkt im endlichen Abstand



- Parallelprojektion:
  Augenpunkt im Unendlichen

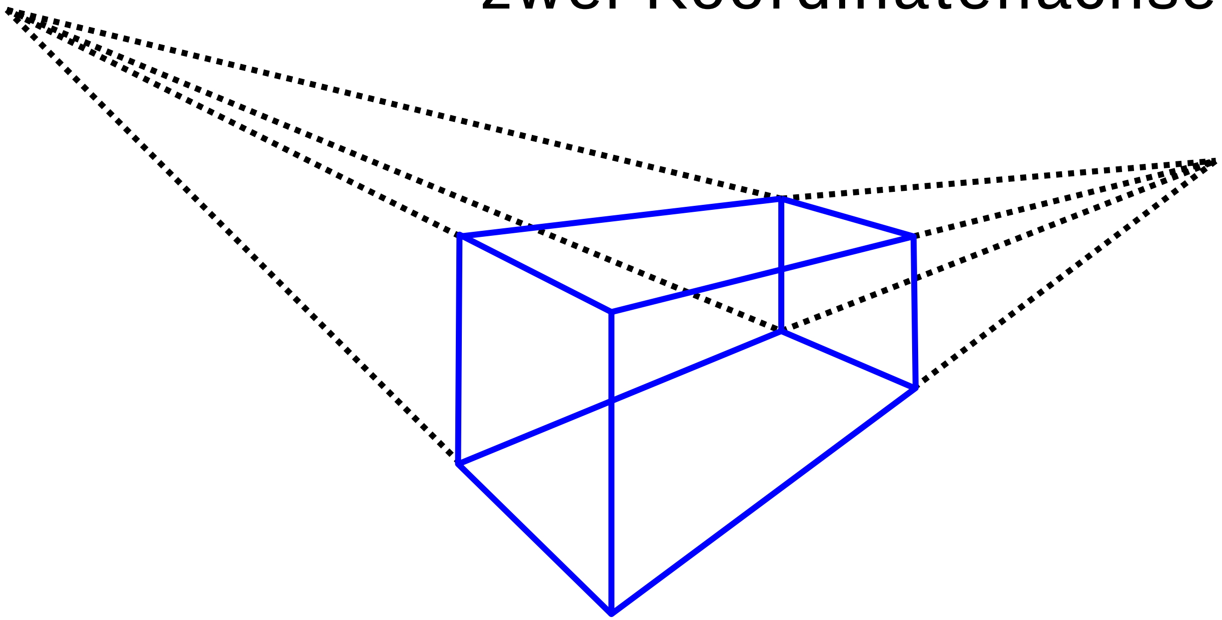
1 Fluchtpunkt



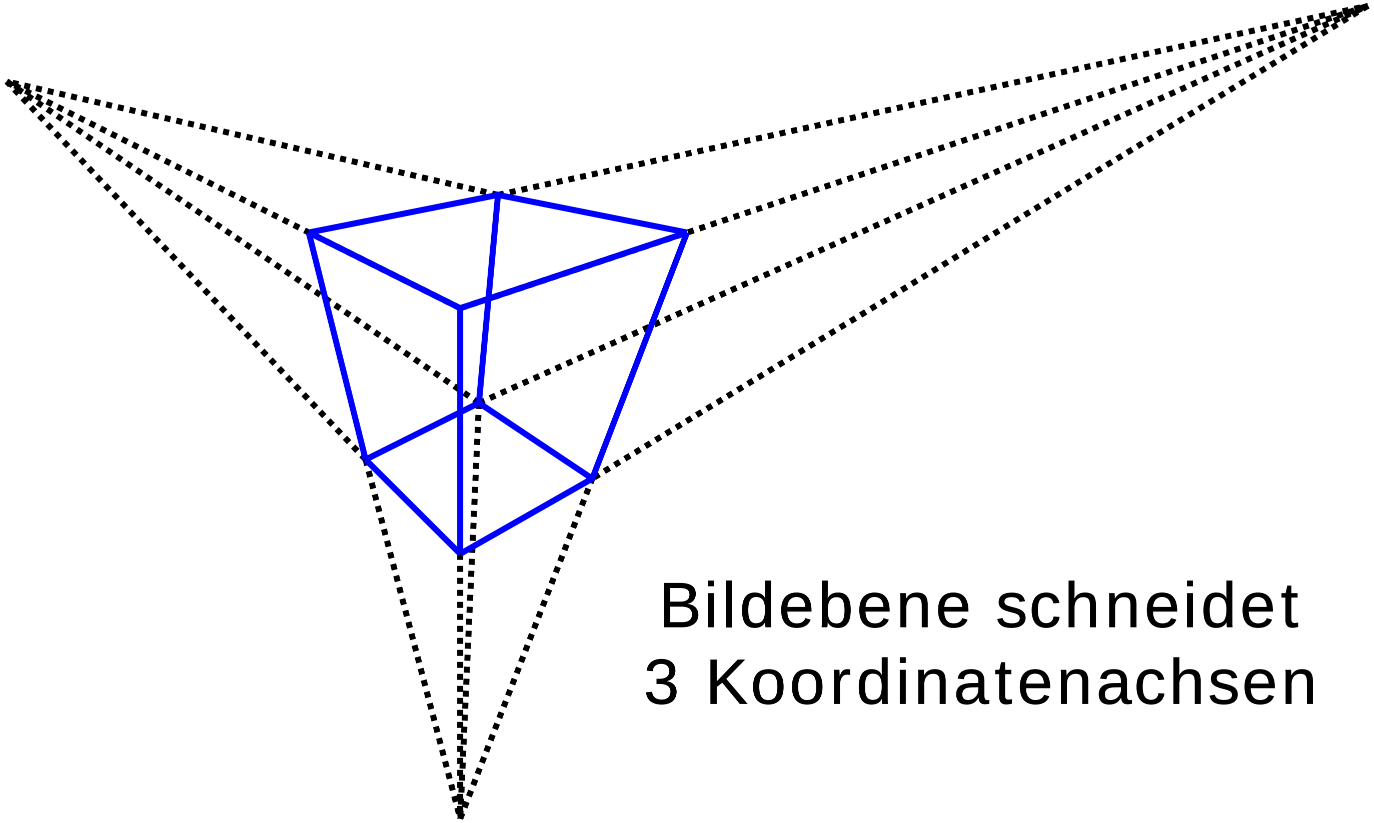
Bildebene schneidet
eine Koordinatenachse

2 Fluchtpunkte

Bildebene schneidet
zwei Koordinatenachsen



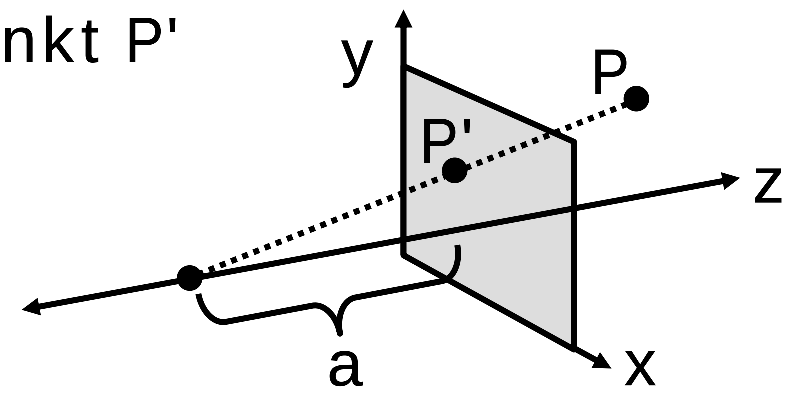
3 Fluchtpunkte



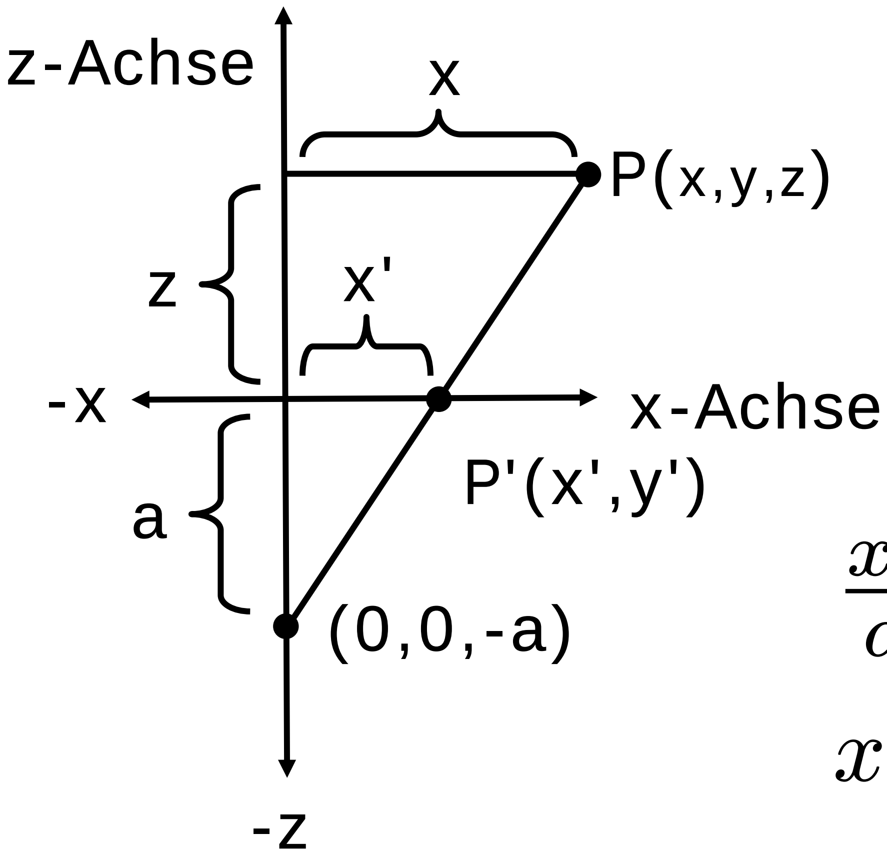
Bildebene schneidet
3 Koordinatenachsen

Aufgabenstellung
(im linkshändigen Koordiantensystem)

- Bildebene sei in xy-Ebene
- Augenpunkt sei auf negativer z-Achse bei -a
- Gegeben Punkt P
- Finde Schnittpunkt P'



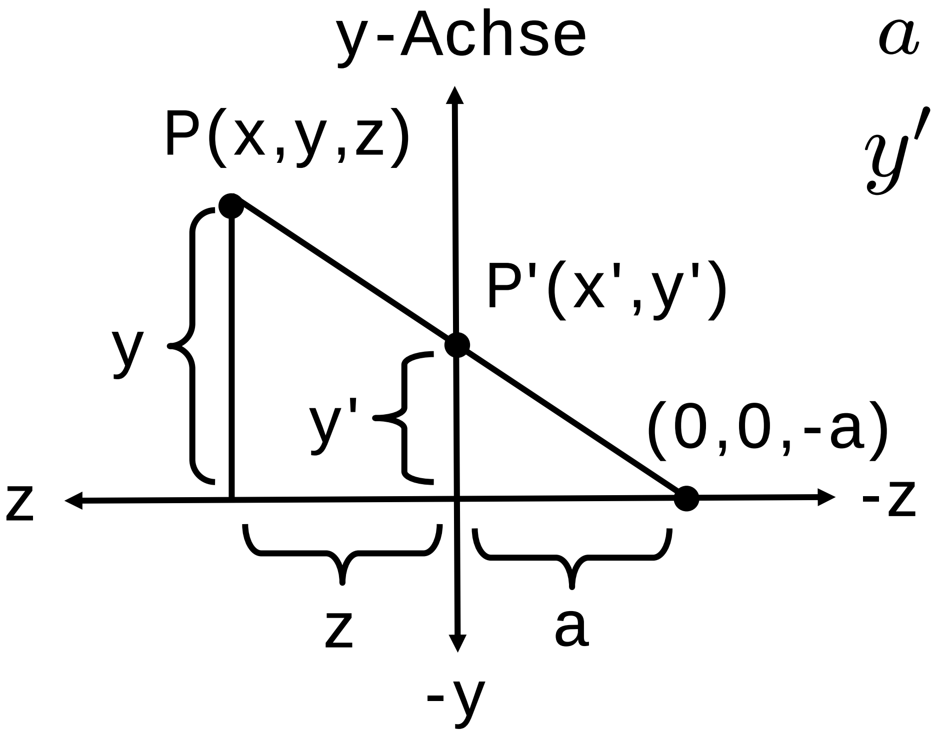
Blick von oben



$$\frac{x'}{a} = \frac{x}{a+z}$$

$$x' = \frac{x}{1+z/a}$$

Blick von der Seite



$$\frac{y'}{a} = \frac{y}{a+z}$$

$$y' = \frac{y}{1+z/a}$$

Ergebnis

$$x' = \frac{x}{1+z/a} \qquad y' = \frac{y}{1+z/a} \qquad z \text{ merken}$$

$$x' = \frac{x}{w} \qquad y' = \frac{y}{w} \qquad w = 1 + z/a$$

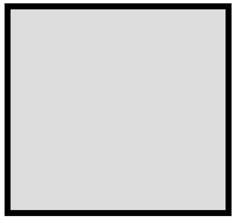
$$P' = (\tfrac{x}{w}, \tfrac{y}{w}, 0, 1) \quad = (x,\ y,\ 0,\ 1 + z/a)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/a & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \\ 1+z/a \end{pmatrix}$$

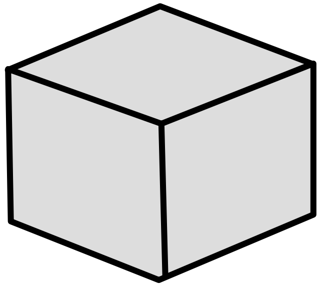
Parallelprojektion

Normalprojektion

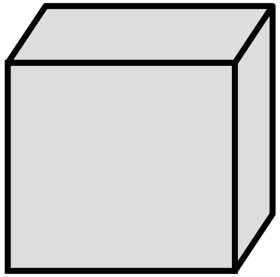
Grund-,Seiten-,Aufriss:



axonometrische
Projektion:



schiefe Projektion

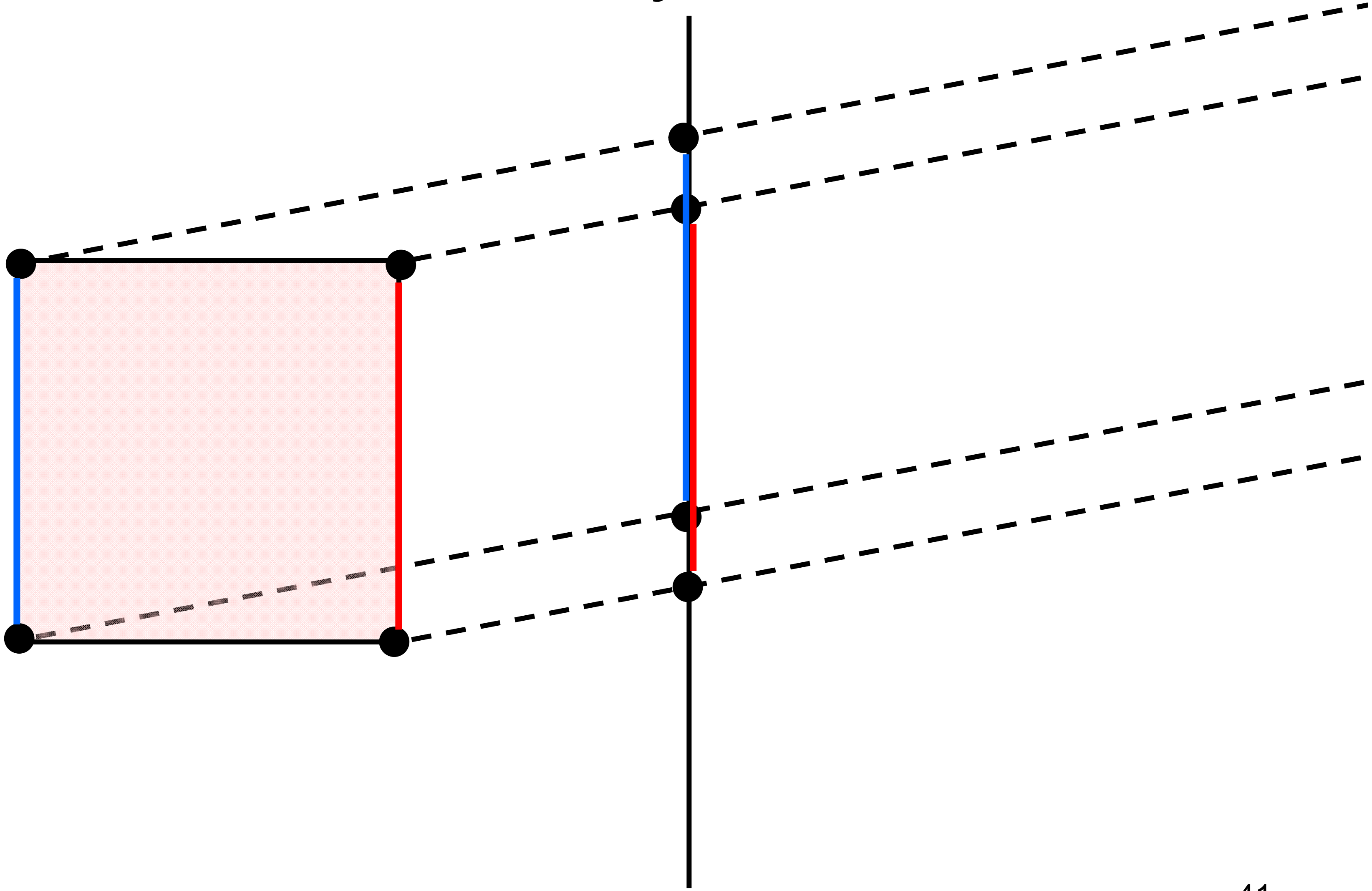


Normalprojektion

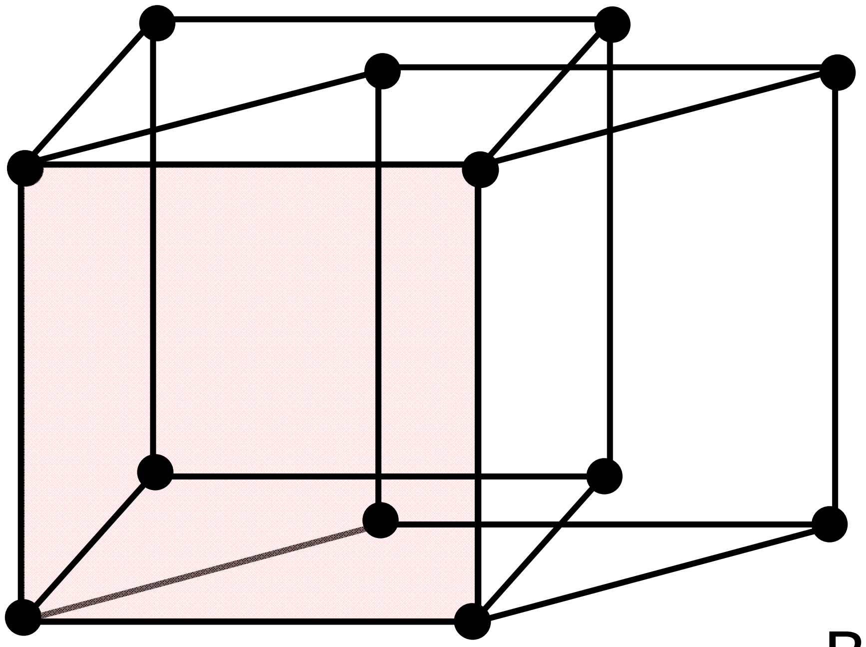
Bilde $(x,y,z,1)$ auf $(x,y,0,1)$ ab:

$$P_{ortho_{xy}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Schiefe Projektion



Schiefe Projektion



Beeinflusst durch ...

...Verkürzung in z-Richtung

... Anstellwinkel

Schiefe Projektion

$$x' = x - L \cdot \cos(\alpha)$$

$$y' = y + L \cdot \sin(\alpha)$$

$$z' = 0$$

$$(x - x')/L = \cos(\alpha)$$

$$(y' - y)/L = \sin(\alpha)$$

$$\tan(\beta) = z/L$$

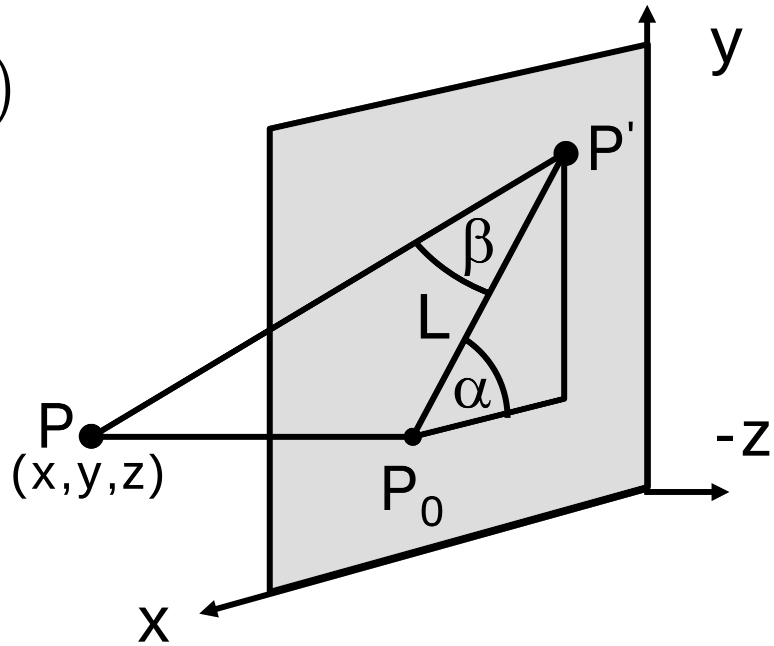
$$L = z/\tan(\beta)$$

$$x' = x - z \cdot (\cos\alpha)/\tan(\beta))$$

$$y' = y + z \cdot (\sin\alpha)/\tan(\beta))$$

α = Anstellwinkel

β = Verkürzungsfaktor



schiefe Transformationsmatrix

$$x' = x - z \cdot \frac{\cos(\alpha)}{\tan(\beta)}$$
$$y' = y + z \cdot \frac{\sin(\alpha)}{\tan(\beta)}$$
$$z' = 0$$
$$w' = 1$$

$$P_{schief_{xy}}(\alpha, \beta) = \begin{pmatrix} 1 & 0 & -\frac{\cos(\alpha)}{\tan(\beta)} & 0 \\ 0 & 1 & \frac{\sin(\alpha)}{\tan(\beta)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

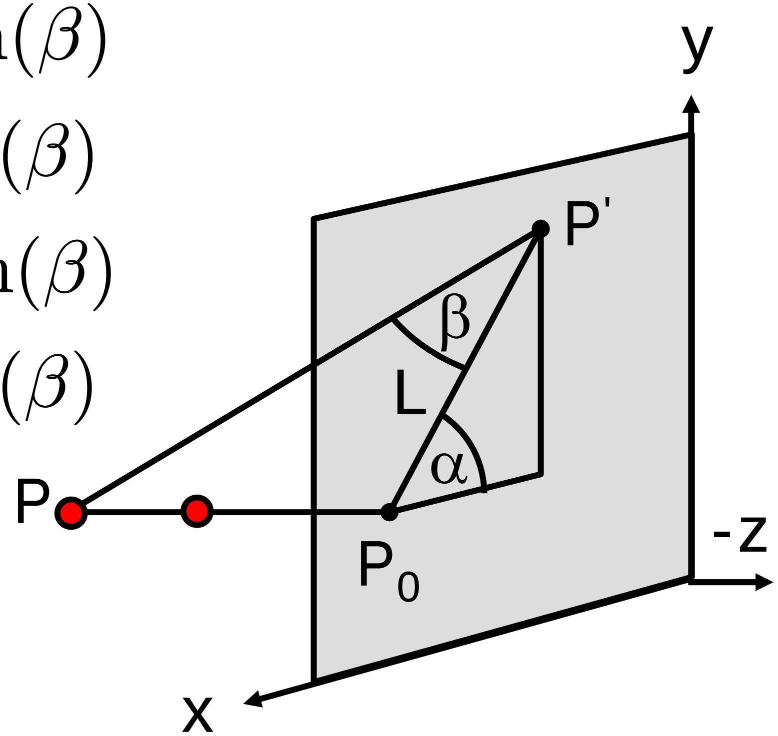
x-Ausdehnung zu z-Ausdehnung

$$x'_1 = x_1 - z_1 \cdot \cos(\alpha)/\tan(\beta)$$

$$y'_1 = y_1 + z_1 \cdot \sin(\alpha)/\tan(\beta)$$

$$x'_2 = x_2 - z_2 \cdot \cos(\alpha)/\tan(\beta)$$

$$y'_2 = y_2 + z_2 \cdot \sin(\alpha)/\tan(\beta)$$



2 Punkte auf Lot zu x/y:

$$|x'_1 - x'_2| = |(z_1 - z_2) \cdot \cos(\alpha)/\tan(\beta)|$$

$$|y'_1 - y'_2| = |(z_1 - z_2) \cdot \sin(\alpha)/\tan(\beta)|$$

Verkürzungsfaktor

$$|P_1' - P_2'| = \sqrt{|x_1' - x_2'|^2 + |y_1' - y_2'|^2}$$

$$|P_1' - P_2'| = \sqrt{\tfrac{(z_1 - z_2)^2}{\tan^2(\beta)} \cdot (\cos^2(\alpha) + \sin^2(\alpha))}$$

$$\cos^2(\alpha) + \sin^2(\alpha) = 1$$

$$= \frac{z_1 - z_2}{\tan(\beta)}$$

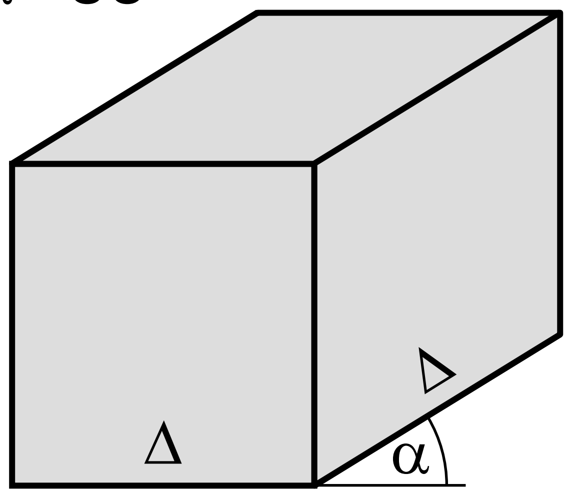
$$d = \frac{1}{\tan(\beta)}$$

$\beta = 45°$ $\Rightarrow d = 1$

$\beta = 63.43°$ $\Rightarrow d = 0.5$

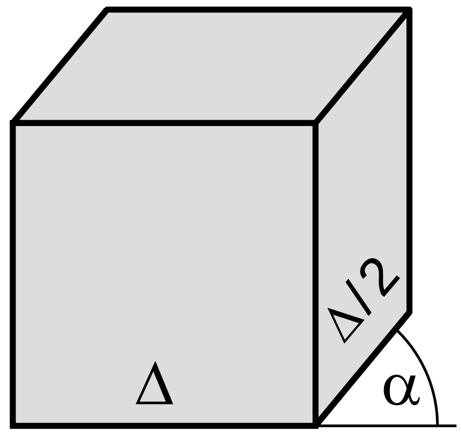
Beispiele für schiefe Projektion

$\beta = 45^\circ \Rightarrow d = 1$
$\alpha = 35^\circ$



Kavalierprojektion

$\beta = 63.43^\circ \Rightarrow d = 0.5$
$\alpha = 50^\circ$



Kabinettprojektion