Theoretical Neuroscience: Exercise 9-1

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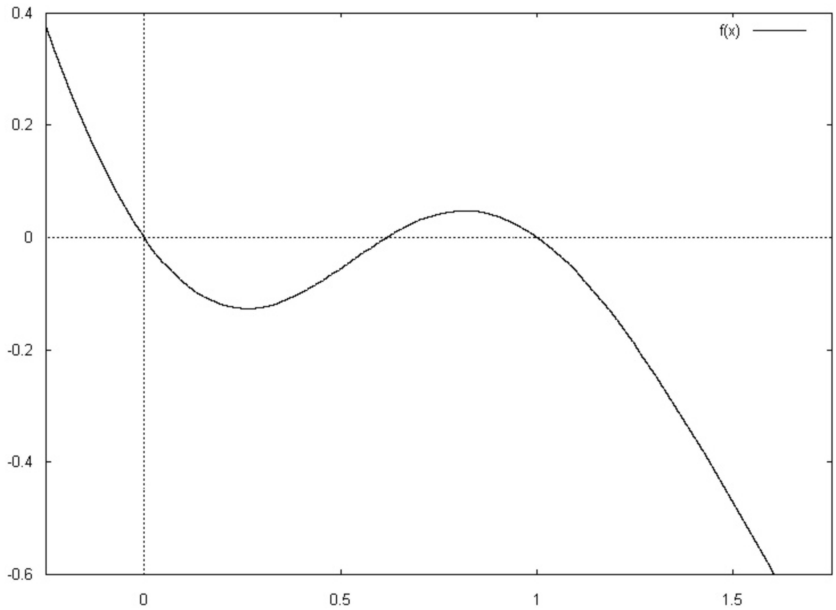
The population density x of a species is determined by the equation

$$\frac{dx}{dt} = \frac{2x^2}{1+x^3} - x \qquad , x \geq 0$$

(a) Graph of dx/dt as a function of x:



(b) The fixed points are determined by calculating the roots of $f(x)$:

$$\begin{aligned} \frac{2x^2}{1+x^3} - x &= 0 \\ 2x^2 &= x + x^4 \\ x^4 - 2x^2 + x &= 0 \\ x(x^3 - 2x + 1) &= 0 \\ x = 0 &\vee x^3 - 2x + 1 = 0 \end{aligned}$$

By looking at the last equation, we see that 1 is a root. Factorizing $(x^3 - 2x + 1) : (x - 1)$ leads to:

$$(x-1)(x^2+x-1)$$

After having solved the quadratic equation, we finally have found 3 roots for $x \geq 0$:

$$x = 0 \ \vee \ x = 1 \ \vee \ x = \frac{1}{2}(\sqrt{5}-1)$$

(c) Stability: To determine the stability we either look at the graph or calculate the slope of $f(x)$ at the fixed points. In the latter case we need the derivative $f'(x)$.

$$f'(x) = \frac{4x-2x^4}{(1+x^3)^2} - 1$$

Slope at the fixed points:

$$\begin{aligned} f'(0) &= -1 \\ f'(\frac{1}{2}(\sqrt{5}-1)) &\approx 0.427051 \\ f'(1) &= -\frac{1}{2} \end{aligned}$$

The first fixed point at $x = 0$ is stable the second one unstable and the fixed point at $x = 1$ is again stable.

(d) Behavior in the limit $t \to \infty$:
With $x(0) = 0.001$ the population decreases to 0, because this is a stable fixed point which is very close.
With $x(0) = 0.8$ the population increases (dx/dt is positive at 0.8) and stabilizes at a density of 1.
A starting population density of 10 will decrease and settle down at a stable density of 1.