Theoretical Neuroscience: Exercise 10-1

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The following ODE describes the growth of the yeast *Schizosaccharomyces kephir*:

$$\frac{dN}{dt} = k_1 N - k_2 N^2 \tag{1}$$

(a) To determine the conditions under which the function

$$N(t) = \frac{a}{1 + be^{-\gamma t}}, \qquad a, b, \gamma > 0 \tag{2}$$

is a solution to the ODE (1), we calculate its derivation

$$N'(t) = \frac{a\gamma b e^{-\gamma t}}{(1 + be^{-\gamma t})^2}. \tag{3}$$

Then we insert (2) into the right hand side of the ODE (1)

$$\begin{aligned} \frac{dN}{dt} &= k_1 \cdot \frac{a}{1 + be^{-\gamma t}} - k_2 \cdot \frac{a^2}{(1 + be^{-\gamma t})^2} \\ &= \frac{k_1 a(1 + be^{-\gamma t}) - k_2 a^2}{(1 + be^{-\gamma t})^2} \\ &= \frac{k_1 a - k_2 a^2 + k_1 a b e^{-\gamma t}}{(1 + be^{-\gamma t})^2} \end{aligned}$$

and determin the conditions under which the result is equal to the derivation (3). This is the obviously the case when $k_1 = \gamma$ and $k_2 = \frac{\gamma}{a}$.

$$\frac{\gamma a b e^{-\gamma t}}{(1 + be^{-\gamma t})^2} = \frac{k_1 a - k_2 a^2 + k_1 a b e^{-\gamma t}}{(1 + be^{-\gamma t})^2}, \qquad k_1 = \gamma,\ k_2 = \frac{\gamma}{a}$$

(b) The fixed points are determined by calculating the roots of (1):

$$\begin{aligned} k_1 N - k_2 N^2 &= 0 \\ N_0 = 0 \quad \vee \quad & k_2 N - k_1 = 0 \\ N_0 = 0 \quad \vee \quad & N_1 = \frac{k_1}{k_2} \end{aligned}$$

The conditions for k_1 and k_2 lead to the fixed points:

$$N_0 = 0 \ \vee \ N_1 = a$$

To determin the fixed points' stabilities we need the derivation of $f(N) = \frac{dN}{dt}$ with respect to N:

$$f'(N) = -2k_2 N + K_1 \tag{4}$$

$$\begin{aligned} f'(0) = & \quad k_1 = \gamma & > 0, \qquad \text{unstable} \\ f'(a) = & \quad -2k_2 a + k_1 = -2\gamma + \gamma = -\gamma & < 0, \qquad \text{stable} \end{aligned}$$

(c) To calculate approximative solutions of the ODE which hold in the neighborhoods of the fixed points, we find the Taylor series[1] of the right hand side of the ODE for each fixed point. For this task we need the ODE (1), its first derivation (4) and its second derivation:

$$f''(N) = -2k_2. \tag{5}$$

We calculate the Taylor series for the neighborhood of $N_0 = 0$

$$\begin{aligned} g_0(N) &= 0 + \frac{k_1}{1!}(N-0) - \frac{2k_2}{2!}(N-0)^2 \\ &= k_1 N - k_2 N^2 \end{aligned}$$

and for the neighborhood of $N_1 = a$.

$$\begin{aligned} g_1(N) &= k_1 a - k_2 a^2 + \frac{k_1 - 2k_2 a}{1!}(N-a) - \frac{2k_2}{2!}(N-a)^2 \\ &= k_1 a - k_2 a^2 - 2k_2 aN + 2k_2 a^2 + k_1 N - k_1 a - k_2 N^2 + 2k_2 aN - k_2 a^2 \\ &= k_1 N - k_2 N^2 \end{aligned}$$

(d) We truncate both Taylor series after the linear term

$$\begin{aligned} h_0(N) &= N_0 + \frac{k_1}{1!}(N - N_0) \\ h_1(N) &= k_1 N_1 - k_2 N_1^2 + \frac{k_1 - 2k_2 N_1}{1!}(N - N_1) \end{aligned}$$

[1]The Taylor series centered around point c is computed by the following formula:

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \dots$$

and introduce a new variable $x \equiv N - N_0$ for h_0 and a new variable $x \equiv N - N_1$ for h_1.

$$\begin{aligned}
h_0(N) &= k_1 x \\
h_1(N) &= k_1 N_1 - k_2 N_1^2 + (k_1 - 2k_2 N_1)x. \\
&= \gamma a - \frac{\gamma}{a} a^2 + \gamma x - \frac{2\gamma}{a} a x \\
&= -\gamma x = -k_1 x
\end{aligned}$$

Now we determin the linear ODE which results for x and solve it.

$$\begin{aligned}
\frac{dx}{dt} &= k_1 x \\
\frac{1}{x} dx &= k_1 dt \\
\int_{x(0)}^{x(t)} \frac{1}{x} dx &= k_1 \int_0^t dt \\
\ln x(t) - \ln x(o) &= k_1 t \\
\ln \frac{x(t)}{x(0)} &= k_1 t \\
x(t) &= x(0) e^{k_1 t}
\end{aligned}$$

The solution for the second ODE which results for x is:

$$x(t) = x(0) e^{-k_1 t}.$$